LECTURE 6

FORK 1003 BI

MATHEMATICS

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Plan:

1. Unconstrained optimization in two variables
2. Lagrange problems

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1 Unconstrained optimization

Problem: max/min $f(x,y)$

Ex: Find max/min $f(x,y) = x^3 - 3xy + y^3$

Methods:

Defn: A stationary pt for f is a point where $f'_x = f'_y = 0$.

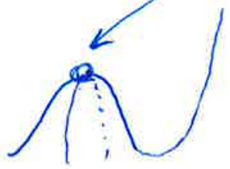
Facts: If (x^*, y^*) is a max/min-pt for f, then it is a stationary point.

if (x^*, y^*) is a stationary pt, we can classify it as
$$\begin{cases} \text{local min} \\ \text{local max} \\ \text{saddle point} \end{cases}$$

Recall: (x^*, y^*) is a local max (local min) for f if

$f(x^*, y^*) \geq f(x,y)$ for all (x,y) close to (x^*, y^*)

$(\leq)$



local max, not global max

Saddle pt: Stationary pt that is not local max or min.

Second derivative test:

If (x^*, y^*) is a stationary pt, we consider

$$H(f)(x^*, y^*) = \begin{pmatrix} f''_{xx}(x^*, y^*) & f''_{xy}(x^*, y^*) \\ f''_{xy}(x^*, y^*) & f''_{yy}(x^*, y^*) \end{pmatrix}$$

$$= \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

Conclusion:

$\det H(f)(x^*, y^*) = AC - B^2 > 0$, $A > 0$: (x^*, y^*) local min

" $AC - B^2 > 0$, $A < 0$: " local max

" $AC - B^2 < 0$ " saddle pt

If $AC - B^2 = 0$, then the test is inconclusive.

Ex: $f = x^3 - 3xy + y^3$

① Compute derivatives:

$$\left.\begin{aligned} f'_x &= 3x^2 - 3y \\ f'_y &= -3x + 3y^2 \end{aligned}\right\} \quad H(f) = \begin{array}{cc} f''_{xx} & f''_{xy} \end{array} \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix} \; f''_{yy}$$

② Find stationary pts:

$f'_x = 0$
$f'_y = 0$

$$\boxed{\begin{aligned} 3x^2 - 3y &= 0 \\ -3x + 3y^2 &= 0 \end{aligned}}$$

FOC = first order conditions.

$\to 3x^2 = 3y$

$\underline{y = x^2}$

$-3x + 3y^2 = 0$

$-3x + 3\cdot(x^2)^2 = 0$

$-3x + 3x^4 = 0$

$3x(-1 + x^3) = 0$

$3x = 0$ or $x^3 - 1 = 0$

$\underline{x = 0}$ $\quad x^3 = 1$

$\underline{y = 0}$ $\quad \underline{x = 1}$ $(\sqrt[3]{1})$

$\underline{y = 1}$

Stationary pts:

$\leftarrow$ $(x,y) = \underline{(0,0), (1,1)}$

③ Classify the stat. pts:

$$H(f) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$$

i) $\underline{(0,0)}$: $H(f)(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$

$|H(f)(0,0)| = AC - B^2 = -9 < 0$

$(0,0)$ is a saddle point

ii) $\underline{(1,1)}$: $H(f)(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$

$|H(f)(1,1)| = AC - B^2 = 36 - 9 = 27 > 0$

$A = 6 > 0 \Rightarrow$ (1,1) is local min

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What about global max/min = max/min

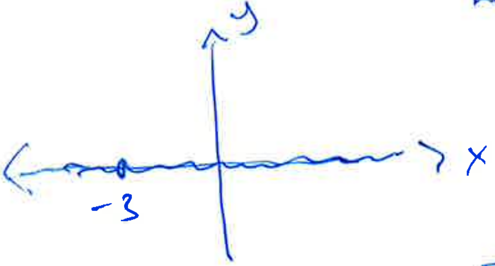
$f = x^3 - 3xy + y^3$ $\quad$ $(0,0)$: saddle pt
$\quad$ $(1,1)$: local min

No global max (since no local max)

Global min: One candidate $(1,1)$, $f(1,1) = -1$

Let $y = 0$, $x = a \to -\infty$

$f(a,0) = a^3 \to -\infty$ when $a \to -\infty$



No global min.

For example: $a = -3$

$f(-3,0) = -27 < -1$

Ex: $f = 1 + x^4 + y^4$

$f'_x = 4x^3 = 0$
$f'_y = 4y^3 = 0$

$$H(f) = \begin{pmatrix} 12x^2 & 0 \\ 0 & 12y^2 \end{pmatrix}$$

with $f''_{xx} = 12x^2$, $f''_{xy} = 0$, $f''_{yy} = 12y^2$

$x = 0$, $y = 0$

Stationary pts:

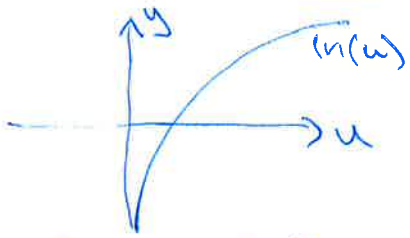
$(x,y) = (0,0)$ $\quad$ $H(f)(0,0) = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$

$|H(f)(0,0)| = \underline{0}$ test inconclusive

$\left.\begin{array}{l} f(0,0) = 1 \\ f(x,y) > 1 \\ \text{for } (x,y) \neq (0,0) \end{array}\right\} \Rightarrow (0,0)$ local min by dfn.
global min

If $f = 1 + x^4 - y^4$ then $(0,0)$ is a saddle pt
test inconclusive

Ex: $f(x,y) = \ln(x^2+y^2+1) = \ln(u), \quad u = x^2+y^2+1$



$$f'_x = \frac{1}{x^2+y^2+1} \cdot 2x = \frac{2x}{u} = 0$$

$$f'_y = \frac{1}{u} \cdot 2y = \frac{2y}{u} = 0$$

Can see that $(0,0)$ is global min

Stationary pts:

$$\frac{2x}{u} = 0, \quad \frac{2y}{u} = 0 \qquad |\cdot u$$

$$2x = 0, \quad 2y = 0$$

$$\cancel{(0,0)} \qquad (x,y) = (0,0)$$

$$f''_{xx} = \frac{2 \cdot u - 2x \cdot 2x}{u^2} = \frac{2(x^2+y^2+1) - 4x^2}{u^2} = \frac{-2x^2+2y^2+2}{u^2}$$

$$f''_{xy} = \frac{0 \cdot u - 2x \cdot 2y}{u^2} = \frac{-4xy}{u^2}$$

$$f''_{yy} = \frac{2 \cdot u - 2y \cdot 2y}{u^2} = \frac{2(x^2+y^2+1) - 4y^2}{u^2} = \frac{2x^2-2y^2+2}{u^2}$$

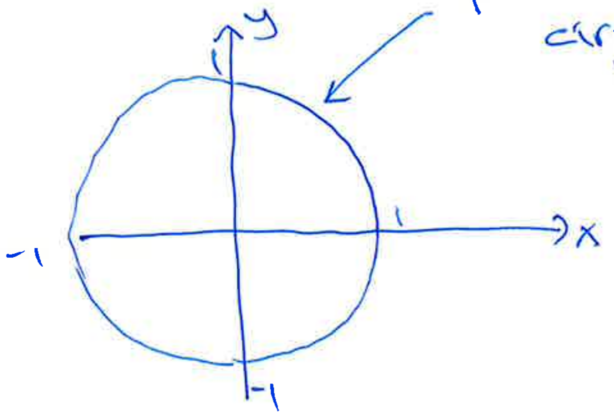
$$H(f)(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \qquad \text{since } u(0,0) = 1$$

$$\left.\begin{array}{l} AC - B^2 = 4 > 0 \\ A = 2 > 0 \end{array}\right\} \quad (0,0) \text{ is local min}$$

② Constrained optimization

Ex: max/min $\overset{f(x,y)}{\overset{\shortparallel}{xy}}$ when $\underbrace{x^2+y^2=1}_{\text{constraint}}$

$\underset{x^2+y^2}{\overset{\shortparallel}{g(x,y)}} = \underset{1}{\overset{\shortparallel}{a}}$



pts with $x^2+y^2=1$
circle w/ centre in $(0,0)$ and radius 1

$\shortparallel$

admissible points

$\longrightarrow (x-x_0)^2+(y-y_0)^2=r^2$

Equation for circle w. centre (x_0,y_0) and radius $r>0$.

Lagrange-problem: max/min $f(x,y)$ when $g(x,y)=a$
(constrained optimization problem with equality constraints)

Lagrangian:

$$\mathcal{L}(x,y;\lambda) = f(x,y) - \lambda \cdot g(x,y)$$

Ex: $\mathcal{L} = xy - \lambda\cdot(x^2+y^2)$

First order conditions (FOC) + **Constraint** (C)

$\mathcal{L}'_x = 0$

$\mathcal{L}'_y = 0$

$g(x,y) = a$

Ex: max/min $f(x,y)=xy$ when $g(x,y)=x^2+y^2=1$

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$$L = xy - \lambda\cdot(x^2+y^2)$$

$$\left.\begin{array}{l} L'_x = y - \lambda\cdot 2x = 0 \\ L'_y = x - \lambda\cdot 2y = 0 \end{array}\right\} \text{FOC} \quad \begin{array}{l}(1)\\(2)\end{array}$$

$$\left.x^2+y^2 = 1\right\} \text{C} \quad (3)$$

(1): $y = 2\lambda x \rightarrow$ (2): $x - 2\lambda\cdot(2\lambda x) = 0$

$x - 4\lambda^2 x = 0$

$x\cdot(1-4\lambda^2) = 0$

$x=0$ or $4\lambda^2=1$

$x=0$	$4\lambda^2=1$
$x=0 \Rightarrow y=0$	$\lambda^2 = 1/4 \Rightarrow \lambda = \pm\sqrt{1/4} = \pm 1/2$
(3) $0=1$ impossible	
no solution	

$\lambda = 1/2$	$\lambda = -1/2$
(1) $y=x$	(1) $y=-x$
(3) $x^2+y^2=1$	(3) $x^2+y^2=1$
$2x^2=1$	$x^2+(-x)^2=1$
$x^2=1/2$	$2x^2=1$
$x=\pm\sqrt{1/2}$	$x^2=1/2$
	$x=\pm\sqrt{1/2}$

Solutions:

$(x,y;\lambda) = \left(\sqrt{1/2}, \sqrt{1/2}; \frac{1}{2}\right)$

$\left(-\sqrt{1/2}, -\sqrt{1/2}; 1/2\right)$

$f=\frac{1}{2}$

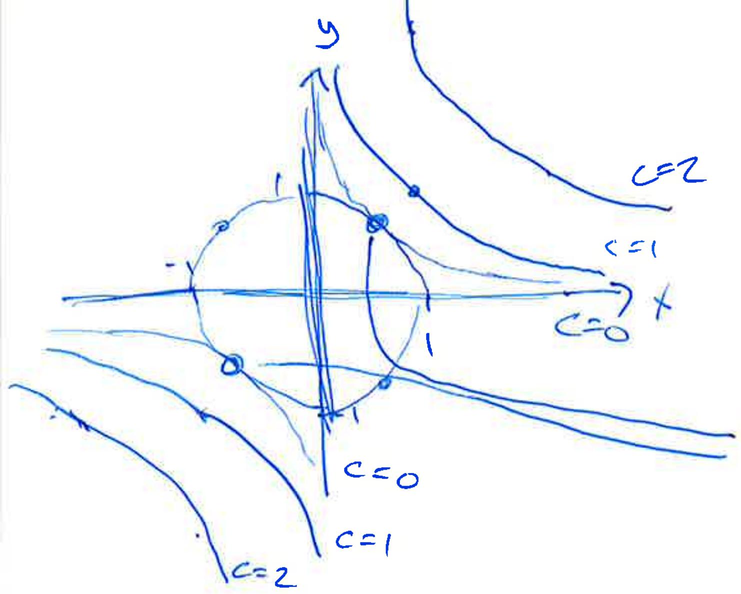
Solutions:

$(x,y;\lambda) = \left(\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}; -\frac{1}{2}\right)$

$\left(-\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}; -\frac{1}{2}\right)$

$f=-\frac{1}{2}$

$\max/\min \ \overset{f}{\overset{\shortparallel}{xy}}$ when $x^2+y^2=1$



max pts :

$\left(\sqrt{\tfrac{1}{2}},\sqrt{\tfrac{1}{2}}\right),\left(-\sqrt{\tfrac{1}{2}},-\sqrt{\tfrac{1}{2}}\right)$

Same tangent at $xy=c$ and $x^2+y^2=1$

$$\begin{pmatrix} f'_x \\ f'_y \end{pmatrix} = \lambda \cdot \begin{pmatrix} g'_x \\ g'_y \end{pmatrix}$$

Level curves for f: $f(x,y)=c$

$xy=c$

$y=c/x$

$$\left.\begin{aligned} f'_x &= \lambda \cdot g'_x \\ f'_y &= \lambda \cdot g'_y \end{aligned}\right\} \text{FOC}$$

From the drawing we can see:

$\left(\sqrt{1/2},\sqrt{1/2}\right),\left(-\sqrt{1/2},-\sqrt{1/2}\right)$ are max : $f=1/2$

$\left(\sqrt{1/2},-\sqrt{1/2}\right),\left(-\sqrt{1/2},\sqrt{1/2}\right)$ min $f=-1/2$

Extreme value theorem:

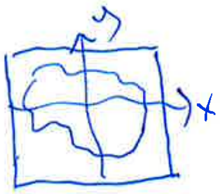
If the set of admissible pts (solution $g(x,y)=a$) is closed and bounded, then

$\max/\min \ f(x,y)$ when $g(x,y)=a$

has a max and min.

For Lagrange-problems, closed is ok (satisfied)

A set is bounded if it contained in a rectangle with finite length and height



Theorem (Necessary condition)

If (x^*, y^*) is a solution to the Lagrange problem

$$\boxed{\max/\min \ f(x,y) \quad \text{when} \quad g(x,y) = a}$$

then either

i) There is a λ s.t. $(x^*, y^*; \lambda)$ solves the FOCs and C.

$$\begin{pmatrix} \mathcal{L}'_x = 0 \\ \mathcal{L}'_y = 0 \end{pmatrix} \qquad g(x,y) = a$$

ii) $$\underbrace{g'_x(x^*, y^*) = g'_y(x^*, y^*) = 0}_{\text{fails NDCQ}} \quad \text{and} \quad g(x,y) = a$$

= non-degenerate constraint qualification

Ex: max y when $x^2 + y^3 = 0$ (remember NDCQ)