FORK1005
Solutions for Exercises 4

July 30, 2015

2 Integration

2.1 Antiderivative

Solution 2.1. The antiderivative of $C'(x)$ is

$$\frac{2}{3}x^3 + x^2 - 5x + K$$

where K is the constant of integration. We have that $C(0) = 100$ so

$$C(0) = K = 100.$$

So the cost function is

$$C(x) = \frac{2}{3}x^3 + x^2 - 5x + 100.$$

Solution 2.2.

(a) $3x^2 + C$

(b) $3x^3 + C$

(c) $\frac{1}{5}x^5 + C$

(d) $-\frac{1}{2x^2} + C$

(e) $\frac{2}{3}x^{3/2} + C$

(f) $\frac{1}{a}e^{ax} + C$

Solution 2.3. If we differentiate $x\ln(x) - x + C$, we get

$$\big(x\ln(x) - x + C\big)' = \ln(x) + \frac{x}{x} - 1 = \ln(x) + 1 - 1 = \ln(x),$$

which is what we wanted to show.

2.2 Integral

Solution 2.4.

(a) $\frac{3}{2}x^2 + \frac{1}{3}x^3 - x^5 + C$

(b) $\frac{1}{3}x^3 + 3x^2 + 9x + C$

2.3 Integration Rules

Solution 2.5.

(a) $3e^x + C$

(b) $4\ln(x) + C$

(c) $\frac{1}{7}x^7 + 2e^x + C$

(d) $-\frac{2}{3x^3} - \ln(x) + C$

Solution 2.6. $3\ln(x) + 2e^{-4x} + C$

3 Integration Techniques

3.1 Integration by Parts

Solution 3.1. We have

$$v = x, \quad v' = 1, \quad u' = e^{4x}, \quad u = \frac{1}{4}e^{4x}$$

so applying integration by parts, we get

$$\begin{aligned}\int xe^{4x}\,\mathrm{d}x &= \frac{x}{4}e^{4x} - \int 1 \cdot \frac{1}{4}e^{4x}\,\mathrm{d}x \\ &= \frac{x}{4}e^{4x} - \frac{1}{16}e^{4x} + C.\end{aligned}$$

Solution 3.2. We have

$$v = 2x, \quad v' = 2, \quad u' = \sqrt{x-1}, \quad u = \frac{2}{3}(x-1)^{3/2}.$$

so applying integration by parts, we get

$$\begin{aligned}\int 2x\sqrt{x-1}\,\mathrm{d}x &= \frac{4x}{3}(x-1)^{3/2} - \int \frac{4}{3}(x-1)^{3/2}\,\mathrm{d}x \\ &= \frac{4x}{3}(x-1)^{3/2} - \frac{8}{15}(x-1)^{5/2} + C.\end{aligned}$$

Solution 3.3.

(a) Choose $u' = e^{-x}$ and $v = x$. Then

$$v = x, \quad v' = 1, \quad u' = e^{-x}, \quad u = -e^{-x}.$$

so applying integration by parts, we get

$$\int xe^{-x}\,\mathrm{d}x = -xe^{-x} + \int e^{-x}\,\mathrm{d}x = -xe^{-x} - e^{-x} + C.$$

(b) Choose $u' = 1$ and $v = \ln(x)$. Then

$$v = \ln(x), \quad v' = \frac{1}{x}, \quad u' = 1, \quad u = x.$$

so applying integration by parts, we get

$$\int \ln(x)\,\mathrm{d}x = \ln(x)x - \int \frac{x}{x}\,\mathrm{d}x = \ln(x) - x + C.$$

3.2 Integration by Substitution

Solution 3.4. $u = x - 4$ so $\frac{\mathrm{d}u}{\mathrm{d}x} = 1$ and $\mathrm{d}x = \mathrm{d}u$. So we have

$$\int (x-4)^6\,\mathrm{d}x = \int u^6\,\mathrm{d}x = \int u^6\,\mathrm{d}u = \frac{1}{7}u^7 + C = \frac{1}{7}(x-4)^7 + C$$

Solution 3.5. $u = x^3 + 13$ so $\frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2$ and $\mathrm{d}x = \frac{1}{3x^2}\,\mathrm{d}u$. So we have

$$\begin{aligned}\int 3x^2(x^3+13)^{20}\,\mathrm{d}x = \int 3x^2u^{20}\,\mathrm{d}x = \int \frac{3x^2}{3x^2}u^{20}\,\mathrm{d}u = \int u^{20}\,\mathrm{d}u &= \frac{1}{21}u^{21} + C \\ &= \frac{1}{21}(x^3+13)^{21} + C.\end{aligned}$$

Solution 3.6. $u = 3 - x$ so $\frac{\mathrm{d}u}{\mathrm{d}x} = -1$ and $\mathrm{d}x = -\,\mathrm{d}u$. So we have

$$\int \frac{1}{3-x}\,\mathrm{d}x = \int \frac{1}{u}\,\mathrm{d}x = \int -\frac{1}{u}\,\mathrm{d}u = -\ln(u) + C = -\ln(3-x) + C.$$

Solution 3.7. $u = \sqrt{1+x^2}$ so $\frac{\mathrm{d}u}{\mathrm{d}x} = \frac{x}{\sqrt{1+x^2}} = \frac{x}{u}$ and $\mathrm{d}x = \frac{u}{x}\,\mathrm{d}u$. So we have

$$\int 2x\sqrt{1+x^2}\,\mathrm{d}x = \int 2xu\,\mathrm{d}x = \int 2xu\frac{u}{x}\,\mathrm{d}u = \int 2u^2\,\mathrm{d}u = \frac{2}{3}u^3 + C = \frac{2}{3}(1+x^2)^{3/2} + C.$$

Solution 3.8.

(a) Set $u = x^2 + 10$. Then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 2x$ and $\mathrm{d}x = \dfrac{1}{2x}\,\mathrm{d}u$. So we have

$$\begin{aligned}\int 2x(x^2+10)^{50}\,\mathrm{d}x = \int u^{50}\,\mathrm{d}x = \int 2xu^{50}\frac{1}{2x}\,\mathrm{d}u = \int u^{50}\,\mathrm{d}u &= \frac{1}{51}u^{51} + C \\ &= \frac{1}{51}(x^2+10)^{51} + C.\end{aligned}$$

(b) Set $u = -cx^2$. Then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = -2cx$ and $\mathrm{d}x = -\dfrac{1}{2cx}\,\mathrm{d}u$. So we have

$$\int xe^{-cx^2}\,\mathrm{d}x = \int xe^u\,\mathrm{d}x = -\int xe^u\frac{1}{2x}\,\mathrm{d}u = -\int \frac{1}{2}e^u\,\mathrm{d}u = -\frac{e^u}{2} + C = -\frac{e^{-cx^2}}{2} + C.$$

(c) Set $u = x^4 - 2x^3 + 5$. Then $\dfrac{\mathrm{d}u}{\mathrm{d}x} = 4x^3 - 6x^2$ and $\mathrm{d}x = \dfrac{1}{4x^3-6x^2}\,\mathrm{d}u$. So we have

$$\begin{aligned}\int (4x^3-6x^2)(x^4-2x^3+5)^7\,\mathrm{d}x &= \int (4x^3-6x^2)u^7\,\mathrm{d}x \\ &= \int \frac{4x^3-6x^2}{4x^3-6x^2}u^7\,\mathrm{d}u \\ &= \int u^7\,\mathrm{d}u \\ &= \frac{1}{8}u^8 + C \\ &= \frac{1}{8}(x^4-2x^3+5)^8 + C.\end{aligned}$$

3.3 Integration by Partial Fractions

Solution 3.9.

(a) We have $x^2 + 2x - 3 = (x+3)(x-1)$. We want A and B such that

$$\frac{A}{x+3} + \frac{B}{x-1} = \frac{Ax - A + Bx + 3B}{(x+3)(x-1)} = \frac{x}{(x+3)(x-1)}$$

This gives us the linear system

$$\begin{aligned} A + B &= 1 \\ -A + 3B &= 0, \end{aligned}$$

which has the unique solution $A = 3/4$, $B = 1/4$. So we have the equality

$$\frac{x}{x^2 + 2x - 3} = \frac{3}{4(x+3)} + \frac{1}{4(x-1)}$$

so

$$\begin{aligned} \int \frac{x}{x^2 + 2x - 3}\,\mathrm{d}x &= \int \frac{3}{4(x+3)} + \frac{1}{4(x-1)}\,\mathrm{d}x \\ &= \frac{3}{4}\ln(x+3) + \frac{1}{4}\ln(x-1) + C. \end{aligned}$$

(b) We have $x^2 - 4x - 12 = (x-6)(x+2)$. We want A and B such that

$$\frac{A}{x-6} + \frac{B}{x+2} = \frac{Ax + 2A + Bx - 6B}{(x-6)(x+2)} = \frac{1+x}{(x-6)(x+2)}$$

This gives us the linear system

$$\begin{aligned} A + B &= 1 \\ 2A - 6B &= 1, \end{aligned}$$

which has the unique solution $A = 7/8$, $B = 1/8$. So we have the equality

$$\frac{1+x}{x^2 - 4x - 12} = \frac{7}{8(x-6)} + \frac{1}{8(x+2)}$$

so

$$\begin{aligned} \int \frac{1+x}{x^2 - 4x - 12}\,\mathrm{d}x &= \int \frac{7}{8(x-6)} + \frac{1}{8(x+2)}\,\mathrm{d}x \\ &= \frac{7}{8}\ln(x-6) + \frac{1}{8}\ln(x+2). \end{aligned}$$

FORK1005
Solutions for Exercises 5

August 10, 2015

2 Partial Differentiation

Solution 2.1.

(a) $f'_x(x,y) = 2, \quad f'_y(x,y) = 4$

(b) $f'_x(x,y) = 3 + 5y, \quad f'_y(x,y) = 5x + 2$

(c) $f'_x(x,y) = 10xy^3, \quad f'_y(x,y) = 15x^2y^2$

(d) $f'_x(x,y) = 3e^xy^2 - 2xe^y, \quad f'_y(x,y) = 6e^xy - x^2e^y$

(e) $f'_x(x,y,z) = 4y^2z^3 - y, \quad f'_y(x,y,z) = 8xyz^3 - x, \quad f'_z(x,y,z) = 12xy^2z^2$

(f) $f'_x(x,y) = ye^{xy} - \dfrac{2y}{x}, \quad f'_y(x,y) = xe^{xy} - \ln(x^2)$

3 First-Order Conditions

Solution 3.1.

(a) $(0,0)$: Saddle point (neither).

(b) $(2,0)$: Minimum.

(c) (x,x) for all x: All points are minima.

(d) $(0,0)$: Maximum.

(e) There are no stationary points.

(f) $(0,0)$: Maximum.

(g) $(1,-2,5)$: Neither.

4 Second-Order Derivatives

4.1 Second-Order Partial Derivatives

Solution 4.1.

(a) $f''_{xx}(x,y) = 2, \quad f''_{xy}(x,y) = 0, \quad f''_{yy}(x,y) = -10$

(b) $f''_{xx}(x,y) = 6, \quad f''_{xy}(x,y) = -4, \quad f''_{yy}(x,y) = -2$

(c) $f''_{xx}(x,y) = 42x^5 - 10y^3, \quad f''_{xy}(x,y) = -30xy^2, \quad f''_{yy}(x,y) = -30x^2y + 12y^2$

(d) $f''_{xx}(x,y) = 24xy, \quad f''_{xy}(x,y) = 12x^2 - e^y, \quad f''_{yy}(x,y) = -xe^y$

(e) $f''_{xx}(x,y) = -\dfrac{y^2}{x^2}, \quad f''_{xy}(x,y) = \dfrac{2y}{x}, \quad f''_{yy}(x,y) = 2\ln(x)$

(f) $f''_{xx}(x,y) = \dfrac{y^2}{(x^2+y^2)^{3/2}}, \quad f''_{xy}(x,y) = -\dfrac{xy}{(x^2+y^2)^{3/2}}, \quad f''_{yy}(x,y) = \dfrac{x^2}{(x^2+y^2)^{3/2}}$

5 Second Partial Derivative Test

Solution 5.1.

(a) Stationary point:

$$(0,0).$$

Hessian matrix:

$$Hf(x,y) = \begin{bmatrix} 6 & 0 \\ 0 & 10 \end{bmatrix}.$$

Determinant:

$$D(x,y) = 6 \cdot 10 - 0 = 60 > 0.$$

We have $D(0,0) > 0$ and $f''_{xx}(0,0) = 6 > 0$ so by the second partial derivative test, $(0,0)$ is a minimum.

(b) Stationary point:

$$(0,0).$$

Hessian matrix:

$$Hf(x,y) = \begin{bmatrix} -2 & 0 \\ 0 & 8 \end{bmatrix}.$$

Determinant:

$$D(x,y) = -2 \cdot 8 - 0 = -16 < 0.$$

We have $D(0,0) < 0$ so by the second partial derivative test, $(0,0)$ is a saddle point.

(c) Stationary point:

$$(0, 0).$$

Hessian matrix:

$$Hf(x, y) = \begin{bmatrix} 6x & 0 \\ 0 & -8 \end{bmatrix}.$$

Determinant:

$$D(x, y) = 6x \cdot (-8) - 0 = -48x.$$

We have $D(0, 0) = 0$ so the results are inconclusive. However, an inspection of the function should convince you that $(0, 0)$ is neither a maximum nor a minimum.

(d) Stationary points:

$$(0, 0) \quad \text{and} \quad (-4/3, 4/3).$$

Hessian matrix:

$$Hf(x, y) = \begin{bmatrix} 6x & -4 \\ -4 & -4 \end{bmatrix}.$$

Determinant:

$$D(x, y) = -24x - 16-$$

We have $D(0, 0) = -16$ so by the second partial derivative test, $(0, 0)$ is a saddle point. We have $D(-4/3, 4/3) = 24(4/3) - 16 = 16 > 0$, and $f''_{xx}(-4/3, 4/3) = -6(4/3) = -8 < 0$ so by the second derivative test, $(-4/3, 4/3)$ is a maximum.

8 Convex vs Concave Functions

Solution 8.1. (a) The Hessian is

$$Hf(x, y) = \begin{bmatrix} 12(x + 2y)^2 & 24(x + 2y)^2 \\ 24(x + 2y)^2 & 48(x + 2y)^2 \end{bmatrix}$$

so

$$D(x, y) = 12 \cdot 48(x + 2y)^4 - 24^2(x + 2y)^4 = 0.$$

Since $f''_{xx}(x, y) = (x + 2y)^4 \geq 0$, f is convex.

(b) The Hessian is

$$Hf(x, y) = \begin{bmatrix} 4(x + y)^2 e^{-(x+y)^2} - 2e^{-(x+y)^2} & 4(x + y)^2 e^{-(x+y)^2} - 2e^{-(x+y)^2} \\ 4(x + y)^2 e^{-(x+y)^2} - 2e^{-(x+y)^2} & 4(x + y)^2 e^{-(x+y)^2} - 2e^{-(x+y)^2} \end{bmatrix}$$

so

$$D(x, y) = 0.$$

We have $f''_{xx}(0, 0) = 0e^0 - 2e^0 = -2 < 0$ and $f''_{xx}(1, 0) = 4e^{-1} - 2e^{-1} = 2/e > 0$, so the function is neither convex or concave.

(c) The Hessian is

$$Hf(x,y) = \begin{bmatrix} 6x+2 & 0 \\ 0 & -2 \end{bmatrix}$$

so

$$D(x,y) = -12x - 4.$$

So $D(0,0) = -4 < 0$ and $D(-1,0) = 8 > 0$, the function is neither convex or concave.

9 Extra Practice Problems

Solution 9.1.

(a) The profit function is given by

$$\begin{aligned} P(x,y) &= xp(x) + yq(y) - C(x,y) \\ &= x(100-4x) + y(80-2y) - x^2 - 3y^2 - 2xy \\ &= -5x^2 - 5y^2 - 2xy + 100x + 80y. \end{aligned}$$

(b) We set P'_x and P'_y equal to zero to get the linear system

$$\begin{cases} x + 5y = 40 \\ 5x + y = 50. \end{cases}$$

This has the unique solution

$$(x,y) = (35/4, 25/4),$$

which is our stationary point. This is a local maximum if $P''_{xx}(35/4, 25/4) < 0$ and $D(35/4, 25/4) > 0$. We have the Hessian matrix

$$Hf(x,y) = \begin{bmatrix} -10 & -2 \\ -2 & -10 \end{bmatrix}$$

so $D(x,y) = 100 - 4 = 96 > 0$, and $P''_{xx}(35/4, 25/4) = -10 < 0$. So by the second partial derivative test, $(35/4, 25/4)$ is a local maximum. Since the profit function P is concave, we conclude that $(35/4, 25/4)$ is a global maximum.

