FORK1003
Solutions for Exercises 1

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1 Introduction to Linear Systems

1.1 Linear Equation

Solution 1.1.

(a) Linear

(b) Non-linear

(c) Non-linear

(d) Linear

(e) Non-linear

(f) Linear

(g) Non-linear

2 Solutions of Linear Systems

Solution 2.1.

(a) $(x_1, x_2) = (4, 3)$

(b) $(x_1, x_2) = (-1/2, 5/2)$

(c) $(x_1, x_2, x_3) = (3, -2, 0)$

(d) $(x_1, x_2, x_3) = (1, -3, 2)$

Solution 2.2.

(a) Infinitely many solutions

(b) No solutions

(c) One solution: $(x_1, x_2) = (11, 1)$

3 Row Reduction

3.1 Coefficient & Augmented Matrix

Solution 3.1.

(a) $\begin{bmatrix} 3 & 2 & 0 \\ 1 & -1 & 1 \\ -2 & -3 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ -4 & 10 & -1 \\ 1 & 0 & 1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2 & -3 & 1 & 0 \\ 0 & 1 & -10 & 8 & -1/2 \end{bmatrix}$

Solution 3.2.

(a) $\left[\begin{array}{cccc|c} 1 & -3 & 8 & -1 & 1 \\ 0 & 0 & 1 & -8 & 13/3 \\ -2 & -1 & 3 & 0 & 0 \end{array}\right]$

(b) $\left[\begin{array}{cccc|c} 6 & 0 & 0 & 0 & 8 \\ 0 & 3 & 0 & 0 & -4 \\ 0 & 0 & -4 & 0 & 2 \\ 0 & 0 & 0 & 18 & 4 \end{array}\right]$

(c) $\left[\begin{array}{ccccc|c} 2 & -7 & -6 & -1 & 0 & 16 \\ 0 & 1 & 11 & -3/2 & -1/2 & 2 \end{array}\right]$

Solution 3.3.

(a) $\begin{cases} 2x_1 + 3x_2 + 4x_3 = 5 \\ x_1 - 2x_2 - 3x_3 = 6 \end{cases}$

(b) $\begin{cases} -2x_1 = 10 \\ 13x_1 + 2x_2 = -16 \\ -3x_1 + 4x_2 = 0 \\ 4x_1 + 2x_2 = 3 \end{cases}$

3.2 Elementary Row Operations

Solution 3.4.

(a)
$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 6 \\ 0 & 3 & 5 & 7 \\ 0 & -1 & 0 & 2 \\ 16 & -6 & 7 & -1 \\ -2 & 4 & 0 & 13 \end{array}\right]$$

(b)
$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 6 \\ -3 & 9 & -4 & -11 \\ -2 & 4 & 0 & 13 \\ 16 & -6 & 7 & -1 \\ 0 & -1 & 0 & 2 \end{array}\right]$$

(c)
$$\left[\begin{array}{ccc|c} 1 & -2 & 3 & 6 \\ 0 & 3 & 5 & 7 \\ -2 & 4 & 0 & 13 \\ -32 & 12 & -14 & 2 \\ 0 & -1 & 0 & 2 \end{array}\right]$$

(d)
$$\left[\begin{array}{cccc|} 1 & 1 & 8 & 13 \\ 0 & 3 & 5 & 7 \\ -2 & 4 & 0 & 13 \\ 16 & -6 & 7 & -1 \\ 0 & -1 & 0 & 2 \end{array}\right]$$

(e)
$$\left[\begin{array}{cccc|} 1 & -2 & 3 & 6 \\ 0 & 3 & 5 & 7 \\ -2 & 4 & 0 & 13 \\ 16 & -6 & 7 & -1 \\ 0 & -1/2 & 0 & 1 \end{array}\right]$$

Solution 3.5.

$$(x_1, x_2) = (3, 2)$$

Possible sequence of row operations:

$$\begin{aligned} R1 &\to \frac{1}{2}R1 \\ R2 &\to R2 - 3R1 \\ R2 &\to \frac{2}{7}R2 \\ R1 &\to R1 + \frac{1}{2}R2 \end{aligned}$$

Solution 3.6.

$$(x_1, x_2, x_3) = (-3, 4, 1)$$

Possible sequence of row operations:

$$\begin{aligned}
&R1 \leftrightarrow R2 \\
&R1 \to \frac{1}{2}R1 \\
&R3 \to R3 - 3R1 \\
&R2 \to \frac{1}{3}R2 \\
&R3 \to R3 - 2R2 \\
&R3 \to \frac{2}{9}R3 \\
&R1 \to R1 + \frac{1}{2}R3 \\
&R2 \to R2 + R3
\end{aligned}$$

3.4 Echelon Forms

Solution 3.7.

(a) Not echelon

(b) Echelon form, not reduced

(c) Not echelon

(d) Echelon and reduced echelon form

3.5 Pivot Positions & Basic Variables

Solution 3.8.

(a)
- Pivot positions: $(1,1),\ (2,2),\ (3,4)$
- Pivot columns: Column 1, 2 and 4
- Basic variables: $x_1,\ x_2,\ x_4$
- Free variables: x_3
- Solution set: $\left\{(x_1, x_2, x_3, x_4) = (x, \frac{1}{2}x + 17, \frac{1}{2}x + 33, 6)\right\}$

(b)
- Pivot positions: $(1,1),\ (2,2),\ (3,3),\ (4,4)$
- Pivot columns: Column 1, 2, 3 and 4
- Basic variables: $x_1,\ x_2,\ x_3,\ x_4$
- Free variables: None
- Solution set: $\{(x_1, x_2, x_3, x_4) = (-38, 10, -5, 0)\}$

Solution 3.9.

(a) We can row reduce it to

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 1/4 & 5/4 \\ 0 & 1 & 0 & -1/12 & -5/12 \\ 0 & 0 & 1 & -11/12 & -7/12 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

for example through the sequence of row operations

$$\begin{aligned} R2 &\to R2 - 3R1 \\ R4 &\to R1 + R4 \\ R2 &\leftrightarrow R3 \\ R2 &\to -R2 \\ R1 &\to R1 - 2R2 \\ R3 &\to R3 + 7R2 \\ R4 &\to R4 - 6R2 \\ R4 &\to R4 + R3 \\ R3 &\to R3/12 \\ R1 &\to R1 + 3R3 \\ R2 &\to R2 - R3 \end{aligned}$$

- Pivot positions: $(1,1),\ (2,2),\ (3,3)$
- Pivot columns: Column 1, 2 and 3
- Basic variables: $x_1,\ x_2,\ x_3$
- Free variables: x_4

(b) We can row reduce it to

$$\left[\begin{array}{cccc|c} 1 & 0 & -4/5 & 2/5 & 3/5 \\ 0 & 1 & 11/5 & -3/5 & 3/5 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

for example through the sequence of row operations

$$
\begin{aligned}
R1 &\to -R1 \\
R2 &\to R2 + 3R1 \\
R3 &\to R3 - 7R1 \\
R4 &\to R4 - 2R1 \\
R2 &\to -R2 \\
R3 &\to R3 - 2R2 \\
R4 &\to R4 - R2 \\
R2 &\to R2/5 \\
R1 &\to R1 + R2
\end{aligned}
$$

- Pivot positions: $(1,1),\ (2,2)$
- Pivot columns: Column 1 and 2
- Basic variables: $x_1,\ x_2$
- Free variables: $x_3,\ x_4$

