

Solutions for Lecture 4

$$1. \quad a) \quad f'(x) = 4x^3 - \frac{3}{2}\sqrt{x} + 3$$

$$f''(x) = 12x^2 - \frac{3}{4\sqrt{x}}$$

$$b) \quad f'(x) = 1 \cdot \sqrt{x^2+1} + x \cdot \frac{1}{2\sqrt{x^2+1}} \cdot 2x = \frac{\sqrt{x^2+1} \cdot \sqrt{x^2+1} + x^2}{\sqrt{x^2+1}}$$
$$= \frac{2x^2+1}{\sqrt{x^2+1}}$$

$$f''(x) = \frac{4x \cdot \sqrt{x^2+1} - (2x^2+1) \cdot \frac{1}{2\sqrt{x^2+1}} \cdot 2x}{x^2+1}$$

$$= \frac{4x(\sqrt{x^2+1}) - x(2x^2+1)}{(x^2+1)\sqrt{x^2+1}}$$

$$= \frac{2x^3+3x}{(x^2+1)^{3/2}}$$

$$c) \quad f'(x) = \left(\frac{x+1}{x^2-3x+2} \right)' = \frac{1 \cdot (x^2-3x+2) - (x+1) \cdot (2x-3)}{(x^2-3x+2)^2}$$

$$= \frac{(x^2-3x+2) - (2x^2-x-3)}{(x^2-3x+2)^2} = \frac{-x^2-2x+5}{(x^2-3x+2)^2}$$

$$f''(x) = \frac{(-2x-2)(x^2-3x+2)^2 - (-x^2-2x+5) \cdot 2(x^2-3x+2) \cdot (2x-3)}{(x^2-3x+2)^4}$$

$$= \frac{(-2x-2)(x^2-3x+2) - (4x-6)(-x^2-2x+5)}{(x^2-3x+2)^3} = \frac{2x^3+6x^2-30x+34}{(x^2-3x+2)^3}$$

$$d) f'(x) = 1 \cdot e^x + x e^x - 2x e^{-x} - x^2 e^{-x} \cdot (-1) + e^{2x-1} \cdot 2$$

$$= (x+1)e^x + (x^2-2x)e^{-x} + 2e^{2x-1}$$

$$f''(x) = (x+2)e^x + (2x-2)e^{-x} + (x^2-2x)e^{-x} \cdot (-1) + 4e^{2x-1}$$

$$= (x+2)e^x + (-x^2+4x-2)e^{-x} + 4e^{2x-1}$$

$$e) f(x) = \ln(x) - \ln(x-1)$$

$$f'(x) = \frac{1}{x} - \frac{1}{x-1} = \frac{x-1-x}{x(x-1)} = \frac{-1}{x(x-1)}$$

$$f''(x) = -\frac{1}{x^2} - \left(-\frac{1}{(x-1)^2}\right) \cdot (1) = -\frac{1}{x^2} + \frac{1}{(x-1)^2} = \frac{(x-1)^2 - x^2}{x^2(x-1)^2}$$

$$= \frac{1-2x}{x^2(x-1)^2}$$

$$f) f'(x) = \frac{1}{x^3-x^2} \cdot (3x^2-2x) = \frac{3x^2-2x}{x^2(x-1)} = \frac{x(3x-2)}{x^2(x-1)} = \frac{3x-2}{x(x-1)}$$

$$f''(x) = \left(\frac{3x-2}{x^2-x} \right)' = \frac{3(x^2-x) - (3x-2) \cdot (2x-1)}{(x^2-x)^2} = \frac{-3x^2+4x-2}{x^2(x-1)^2}$$

2. a) $f'(x) = 2x$ $f''(x) = 2 > 0$ for all $x \rightarrow$ convex

b) $f'(x) = ax^{a-1}$ $f''(x) = a \cdot (a-1)x^{a-2}$

Since $a > 0$, $x^{a-2} > 0$,
we have that

$$f''(x) > 0 \text{ when } a > 1 \rightarrow a > 1: f \text{ convex}$$

$$f''(x) < 0 \text{ " } a < 1 \rightarrow 0 < a < 1: f \text{ concave}$$

c) $f' = f'' = e^x > 0 \rightarrow$ convex

d) $f' = \ln x$ $f'' = \frac{1}{x} = x^{-1}$ $f''' = -\frac{1}{x^2} < 0 \rightarrow$ concave

e) $f'' = \frac{-3x^2+4x-2}{x^2(x-1)^2}$ by 1f), and $x^2, (x-1)^2 \geq 0$.

$$-3x^2+4x-2=0$$

$$x = \frac{-4 \pm \sqrt{16-4 \cdot 6}}{2 \cdot (-3)}$$

no real
solution

$\rightarrow$ no intersection
with x -axis
 $f''(x) < 0$ for all $x > 1$

$\rightarrow$ concave