HANDELSHØYSKOLEN

Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo

Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500

www.bi.no
info@bi.no
Org. nr. 971 22 8865

LECTURE 1

FORK 1003

EIVIND ERIKSEN

JUL 31ST, 2017

MATHEMATICS

Plan:

1. Solving linear systems
2. Gaussian elimination

Textbook:

Simon, Blume [ME] Mathematics for economists.

1

Problem:

Solve the following systems:

a)
$$\begin{aligned} 7x - 3y &= 6 \\ 2x + y &= 11 \end{aligned}$$

b)
$$\begin{aligned} x + y - 2z + 3w &= 2 \\ 2x - y + z + 5w &= 3 \\ x + 7y + 10z + 5w &= 4 \end{aligned}$$

Methods:

- substitution methods
- elimination methods

a)
$$\boxed{\begin{aligned} 7x - 3y &= 6 \\ 2x + y &= 11 \end{aligned}}$$

Substitution:

① $7x - 3y = 6$

$-3y = 6 - 7x$

$y = \frac{6-7x}{-3}$

$= -2 + \frac{7}{3}x$

$$y = \frac{7}{3}x - 2$$

② $2x + y = 11$

$2x + \left(\frac{7}{3}x - 2\right) = 11 \quad | \cdot 3$

$6x + 7x - 6 = 33$

$\frac{13x}{13} = \frac{39}{13}$

$\underline{x = 3}$

$y = \frac{7}{3} \cdot 3 - 2 = \underline{5}$

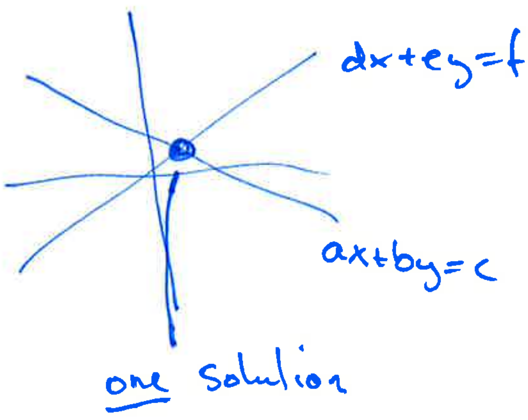
Solution: $(x, y) = \underline{(3, 5)}$

Notice: For 2×2 - linear systems

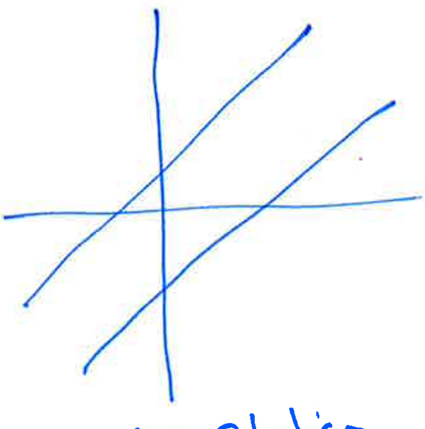
$$\boxed{\begin{aligned} ax + by &= c \\ dx + ey &= f \end{aligned}}$$

each equation is a straight line, so the system is either:

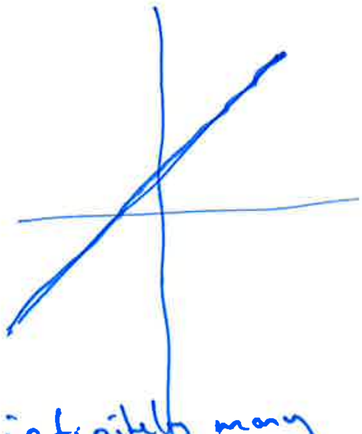
solutions = intersection pts



one solution



no solutions



infinitely many solutions

BI HANDELSHØYSKOLEN
Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo
Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500
www.bi.no
info@bi.no
Org. nr. 971 22 8865

Elimination:

$$7x - 3y = 6$$
$$2x + y = 11 \quad 3\cdot|$$

add:
$$\begin{cases} 7x - 3y = 6 \\ 6x + 3y = 33 \end{cases}$$
$$13x = 39$$

$$x = \frac{39}{13} = \underline{3}$$

$$2\cdot 3 + y = 11$$
$$y = 11 - 6$$
$$\underline{y = 5}$$

Solution: $(x,y) = \underline{\underline{(3,5)}}$

Notice: Operate on the system, which can be repr. by a matrix:

$$\left(\begin{array}{cc|c} 7 & -3 & 6 \\ 2 & 1 & 11 \end{array}\right)$$

Eliminate variables to simplify system.

Defn:

Linear system:

A linear equation in $x_1 \dots x_n$ is an equation with terms of degree one or degree zero (constants) and has the form

$$\boxed{a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b}$$

where $a_1, \dots, a_n$, b are given numbers.

A linear system in $x_1, \dots, x_n$ is a collection of m linear equation in $x_1, \dots, x_n$. It is called an $m \times n$ linear system and has the form

$$\begin{aligned}
a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n &= b_1 \\
a_{21} x_1 + a_{22} x_2 + \dots + a_{2n} x_n &= b_2 \\
&\vdots \\
a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n &= b_m
\end{aligned}$$

where $a_{11}, \dots a_{mn}$, $b_1, \dots, b_m$ are given numbers.

BI HANDELSHØYSKOLEN
Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo
Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500
www.bi.no
info@bi.no
Org. nr. 971 22 8865

② Gaussian elimination

General method for solving any linear systems, it is efficient, and it is instructional.

$$m\left\{\boxed{\begin{array}{l} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{array}}\right.$$

general $m \times n$ linear system

n

augmented coefficient matrix of the linear system

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

$n \qquad 1$

$m \times (n+1)$-matrix

m rows

$n+1$ columns

Ex:

$$\begin{aligned} x + 3y - z &= -2 \\ 2x - y + z &= 3 \\ x + y + z &= 6 \end{aligned}$$

3x3 linear system

$\downarrow$ augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 3 & -1 & -2 \\ 2 & -1 & 1 & 3 \\ 1 & 1 & 1 & 6 \end{array}\right)$$

BI HANDELSHØYSKOLEN
Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo
Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500
www.bi.no
info@bi.no
Org. nr. 971 22 8865

Echelon form:

Ex:

$$\left(\begin{array}{ccc|c} 3 & -1 & 2 & -1 \\ 0 & 4 & 2 & 12 \\ 0 & 0 & -1 & 3 \end{array}\right)$$

is an echelon form

Defn:

A leading coefficient (= pivot) (in a row) is the first non-zero number in the given row (starting from the left).

A matrix is in echelon form if the following conditions are met:

1. All rows with only zeros (zero rows) are in the bottom of the matrix.
2. Any leading coefficient is further to the right than the leading coefficients in rows above.

In practice, echelon forms have zeros under each leading coefficient.

Ex:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 3 \\ 0 & 0 & -1 & 4 \\ 0 & 0 & 0 & 7 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 0 & 0 & 1 & 4 \\ 0 & 3 & -1 & 0 \\ 1 & 1 & 2 & 1 \end{array}\right)$$

Why do we want to get to an echelon form?

Ex:

$$\left(\begin{array}{ccc|c} 3 & -1 & 2 & -1 \\ 0 & 4 & 2 & 12 \\ 0 & 0 & -1 & 3 \end{array}\right) \qquad \begin{aligned} 3x - y + 2z &= -1 \\ 4y + 2z &= 12 \\ -z &= 3 \end{aligned}$$

Backward substitution: - start from the last equation and move towards the top

$$-z = 3$$
$$z = -3$$

$$4y + 2z = 12$$
$$4y + 2\cdot(-3) = 12$$
$$4y = 12 + 6 = 18$$
$$y = 18/4 = 9/2$$

$$3x - y + 2z = -1$$
$$3x - \frac{9}{2} + 2(-3) = -1$$
$$3x = -1 + 6 + 9/2$$
$$3x = 19/2$$
$$x = 19/6$$

Solution: $(x,y,z) = \left(\frac{19}{6}, \frac{9}{2}, -3\right)$

Gaussian elimination

linear system → augmented matrix

↓ elementary row operations

echelon form → linear system

solve by backward substitution

Elementary row operations

Ex:

$$\begin{aligned} x + 3y - z &= -2 \\ 2x - y + z &= 3 \\ x + y + z &= 6 \end{aligned} \quad \rightsquigarrow \quad \left(\begin{array}{ccc|c} 1 & 3 & -1 & -2 \\ 2 & -1 & 1 & 3 \\ 1 & 1 & 1 & 6 \end{array}\right)$$

BI HANDELSHØYSKOLEN
Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo
Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500
www.bi.no
info@bi.no
Org. nr. 971 22 8865

Elementary row operations:

1. Interchange two rows
2. Multiply a row with $c \neq 0$ $\quad \cdot c$
3. Add c times row i to row j.

$$R(j) := R(j) + c \cdot R(i)$$

Ex:

$$\left(\begin{array}{ccc|c} 0 & 1 & 3 & 4 \\ 1 & 2 & 3 & -1 \\ 0 & 0 & 1 & 7 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 2 & 3 & -1 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 7 \end{array}\right)$$

$$\frac{1}{2}\left(\begin{array}{ccc|c} 2 & 3 & -1 & 7 \\ -1 & 4 & 2 & 6 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \cdot 2 \to \left(\begin{array}{ccc|c} 2 & 3 & -1 & 7 \\ -2 & 8 & 4 & 12 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

Alt.

$$\downarrow \qquad\qquad\qquad\qquad \downarrow$$

$$\left(\begin{array}{ccc|c} 2 & 3 & -1 & 7 \\ 0 & 11/2 & 3/2 & 19/2 \\ \hline 0 & 0 & 0 & 1 \end{array}\right) \qquad \left(\begin{array}{ccc|c} 2 & 3 & -1 & 7 \\ 0 & 11 & 3 & 19 \\ 0 & 0 & 0 & 1 \end{array}\right)$$

echelon form $\qquad\qquad$ echelon form

BI HANDELSHØYSKOLEN
Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo
Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500
www.bi.no
info@bi.no
Org. nr. 971 22 8865

Key result:

* Any matrix can be transformed into an echelon form using elementary row operations.
* The echelon form is not unique, but the pivot positions (the positions in the echelon form with pivots) are.

BI HANDELSHØYSKOLEN

Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo

Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500

www.bi.no
info@bi.no
Org. nr. 971 22 8865

Ex:

$$\begin{aligned} x + 3y - z &= -2 \\ 2x - y + z &= 3 \\ x + y + z &= 6 \end{aligned}$$

get zeros

Strategy: use the pivot

$$\left(\begin{array}{ccc|c} 1 & 3 & -1 & -2 \\ 2 & -1 & 1 & 3 \\ 1 & 1 & 1 & 6 \end{array}\right) -2 \rightarrow \left(\begin{array}{ccc|c} 1 & 3 & -1 & -2 \\ 0 & -7 & 3 & 7 \\ 1 & 1 & 1 & 6 \end{array}\right) -1$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 3 & -1 & -2 \\ 0 & -7 & 3 & 7 \\ 0 & -2 & 2 & 8 \end{array}\right) -\frac{2}{7} \rightarrow \left(\begin{array}{ccc|c} 1 & 3 & -1 & -2 \\ 0 & -7 & 3 & 7 \\ 0 & 0 & 8/7 & 6 \end{array}\right)$$

echelon form

get zero

Strategy: use pivot

$$3 \cdot \left(-\frac{2}{7}\right) + 2 = \frac{-6}{7} + \frac{14}{7} = 8/7$$

Alt:

$$\left(\begin{array}{ccc|c} 1 & 3 & -1 & -2 \\ 0 & -2 & 2 & 8 \\ 0 & -7 & 3 & 7 \end{array}\right) \cdot 1/2 \rightarrow \left(\begin{array}{ccc|c} 1 & 3 & -1 & -2 \\ 0 & -1 & 1 & 4 \\ 0 & -7 & 3 & 7 \end{array}\right) -7$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 3 & -1 & -2 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & -4 & -21 \end{array}\right)$$

BI HANDELSHØYSKOLEN
Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo
Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500
www.bi.no
info@bi.no
Org. nr. 971 22 8865

Solutions:

$$\left(\begin{array}{cccc|c} 1 & 2 & 3 & -1 & 1 \\ 0 & 0 & 1 & 4 & 7 \end{array}\right) \qquad \begin{aligned} \underline{x} + 2y + 3z - w &= 1 \\ \underline{z} + 4w &= 7 \end{aligned}$$

echelon form

backwards substitution

$$z + 4w = 7$$

$$z = \underline{7 - 4w}$$

$$x + 2y + 3z - w = 1$$

$$x + 2y + 3(7-4w) - w = 1$$

$$x = 1 - 2y - 21 + 13w$$

$$= \underline{-20 - 2y + 13w}$$

x, z: basic variables ← pivot positions in x- and z-col's.

y, w: free variables ← not pivot positions in y- and w-col's

Solution:

$$\boxed{\begin{aligned} x &= -20 - 2y + 13w \\ y &= y \quad (\text{free}) \\ z &= 7 - 4w \\ w &= w \quad (\text{free}) \end{aligned}}$$

infinitely many solutions

Ex: $y = 1$, $w = 4$ $\qquad \underline{x = 30} \quad \underline{y = 1} \quad z = \underline{-9} \quad \underline{w = 4}$

BI HANDELSHØYSKOLEN
Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo
Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500
www.bi.no
info@bi.no
Org. nr. 971 22 8865

Ex:

$$\left(\begin{array}{ccc|c} 1 & 3 & 5 & 7 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{array}\right) \qquad 0\cdot x + 0\cdot y + 0\cdot z = 4$$

pivot in last (augmented) column

$\Updownarrow$

no solutions

Conclusion:

Any linear system has either

i) one unique solution

ii) infinitely many solution

iii) no solutions

BI HANDELSHØYSKOLEN
Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo
Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500
www.bi.no
info@bi.no
Org. nr. 971 22 8865

Problems:

[ME] 7.1, 7.2, 7.3, 7.12, 7.13, 7.14, 7.15, 7.16,

Solution of b) from page 1 of the lecture notes:

$$\begin{aligned} x + y - 2z + 3w &= 2 \\ 2x - y + z + 5w &= 3 \\ x + 7y + 10z + 5w &= 4 \end{aligned} \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 2 \\ 2 & -1 & 1 & 5 & 3 \\ 1 & 7 & 10 & 5 & 4 \end{array}\right) \begin{matrix} -2 \\ \\ -1 \end{matrix}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 2 \\ 0 & -3 & 5 & -1 & -1 \\ 0 & 6 & 12 & 2 & 2 \end{array}\right) \begin{matrix} \\ 2 \\ \end{matrix} \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & -2 & 3 & 2 \\ 0 & -3 & 5 & -1 & -1 \\ 0 & 0 & 22 & 0 & 0 \end{array}\right)$$

echelon form

x, y, z : basic

w : free

Infinitely many solutions

(all basic var's can be expressed in terms of the free)

$$22z = 0 \Downarrow z = 0$$

$$-3y + 5z - w = -1$$
$$-3y + 5\cdot 0 - w = -1$$
$$-3y = w - 1$$
$$y = -\frac{1}{3}w + \frac{1}{3}$$

$$x + y - 2z + 3w = 2$$
$$x + \left(-\frac{1}{3}w + \frac{1}{3}\right) - 2\cdot 0 + 3w = 2$$
$$x = 2 - \frac{8}{3}w - \frac{1}{3}$$
$$x = \frac{5}{3} - \frac{8}{3}w$$

Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo
Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500
www.bi.no
info@bi.no
Org. nr. 971 22 8865

Solution:

$$x = -\frac{8}{3}w + \frac{5}{3}$$

$$y = -\frac{1}{3}w + \frac{1}{3}$$

$$z = 0$$

$$w = w \quad \text{(free)}$$