BI HANDELSHØYSKOLEN
Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo
Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500
www.bi.no
info@bi.no
Org. nr. 971 22 8865

LECTURE 2

FORK 1003

EIVIND ERIKSEN, AUG 01, 2017

MATHEMATICS

Plan:

1. Matrices
2. Vectors
3. Matrix algebra and inverse matrices

Review: Gaussian elimination

7.12 in [ME]:

$$\begin{aligned} w + x + 3y - 2z &= 0 \\ 2w + 3x + 7y - 2z &= 9 \\ 3w + 5x + 13y - 9z &= 1 \\ -2w + x \qquad\quad - z &= 0 \end{aligned} \rightsquigarrow \left(\begin{array}{cccc|c} 1 & 1 & 3 & -2 & 0 \\ 2 & 3 & 7 & -2 & 9 \\ 3 & 5 & 13 & -9 & 1 \\ -2 & 1 & 0 & -1 & 0 \end{array}\right) \begin{array}{l} -2 \\ -3 \\ 2 \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 3 & -2 & 0 \\ 0 & 1 & 1 & 2 & 9 \\ 0 & 2 & 4 & -3 & 1 \\ 0 & 3 & 6 & -5 & 0 \end{array}\right) \begin{array}{l} -2 \\ -3 \end{array} \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 3 & -2 & 0 \\ 0 & 1 & 1 & 2 & 9 \\ 0 & 0 & 2 & -7 & -17 \\ 0 & 0 & 3 & -11 & -27 \end{array}\right) \; -\frac{3}{2}$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 3 & -2 & 0 \\ 0 & 1 & 1 & 2 & 9 \\ 0 & 0 & 2 & -7 & -17 \\ 0 & 0 & 0 & -1/2 & -3/2 \end{array}\right) \cdot 2 \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 3 & -2 & 0 \\ 0 & 1 & 1 & 2 & 9 \\ 0 & 0 & 2 & -7 & -17 \\ 0 & 0 & 0 & -1 & -3 \end{array}\right)$$

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Remark: Gauss-Jordan elimination

Defn: A reduced echelon form is an echelon form where the following additional conditions are met:

1. All pivots are equal to 1.
2. All entries above a pivot are zero.

Any matrix can be transformed into a reduced echelon form using elementary row operations. The reduced echelon form is unique.

Gauss-Jordan elimination:

Same as Gauss elimination, but you continue until you find a reduced echelon form.

$$\begin{aligned} w + x + 3y - 2z &= 0 \\ x + y + 2z &= 9 \\ 2y - 7z &= -17 \\ -z &= -3 \end{aligned}$$

$w = 0 - 1 - 6 + 6 \quad w = -1$

$x = 9 - 2 - 6 \quad x = 1$

$2y = -17 + 21 = 4 \quad y = 2$

$z = 3$

Sol: $(w, x, y, z) = (-1, 1, 2, 3)$

Reduced echelon form:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 3 \end{array}\right)$$

$$\begin{aligned} w \qquad\qquad &= -1 \\ x \qquad &= 1 \\ y \quad &= 2 \\ z &= 3 \end{aligned}$$

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① Matrices

Defn: An $m \times n$-*matrix* A is a rectangular array of numbers with m rows, n columns:

$$A = \underbrace{\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}}_{n \text{ columns}} \left.\vphantom{\begin{matrix} a \\ a \\ a \\ a \end{matrix}}\right\} m \text{ rows}$$

Ex: $A = \begin{pmatrix} 2 & 3 & 1 \\ -1 & 1 & 0 \end{pmatrix}$ $\qquad a_{13} = 1$

2×3 matrix $\qquad$ row column

Operations on matrices:

Ⓐ **Addition/subtraction:** possible if the matrices have the same *size*.

'position by position'

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 6 & 5 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 3 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 \end{pmatrix} + \begin{pmatrix} 2 & 3 & 7 \\ -1 & 4 & 1 \end{pmatrix} \quad \text{not defined}$$

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(B) Scalar multiplication : $r \cdot A$

Ex: $\underset{\substack{\uparrow \\ \text{number} \\ \text{(scalar)}}}{\overset{r=}{2}} \cdot \underset{\substack{\uparrow \\ \text{matrix}}}{\overset{A=}{\begin{pmatrix} 1 & 0 & -1 \\ 0 & 3 & 4 \end{pmatrix}}} = \underline{\underline{\begin{pmatrix} 2 & 0 & -2 \\ 0 & 6 & 8 \end{pmatrix}}}$

(2) Vectors

Defn: A vector (column vector, m-vector) is a matrix with one column ($m \times 1$ matrix)

Ex: $\underline{v} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ $\quad \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$

$x=1,\ y=2,\ z=3$

Special case of a matrix

Ex: $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} + \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -2 \end{pmatrix}$ $\quad 2 \cdot \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}$

$\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix} = \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix}$

c) Matrix multiplication: $A \cdot B$

Ex:

$$\begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix} \cdot \begin{pmatrix} 4 & 1 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 1\cdot4+0\cdot0 & 1\cdot1+0\cdot2 \\ 3\cdot4+(-1)\cdot0 & 3\cdot1+(-1)\cdot2 \end{pmatrix} = \begin{pmatrix} 4 & 1 \\ 12 & 1 \end{pmatrix}$$

$$2\times2 = 2\times2$$

Note:

1. For $A\cdot B$ to be defined, the # col's in A = # row's in B

   If A is $m\times n$ and B is $n\times p$, then $A\cdot B$ is an $m\times p$-matrix.

2. $(AB)_{ij} =$ ~~RowxA~~ $\leftarrow$ row i of A, col. j of B

$$= a_{i1}\cdot b_{1j} + a_{i2}\cdot b_{2j} + \dots + a_{in}\cdot b_{nj}$$

Ex:

$$\underset{2\times2}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}} \cdot \underset{2\times1}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} = \begin{pmatrix} 1\cdot1+2\cdot2 \\ 3\cdot1+4\cdot2 \end{pmatrix} = \begin{pmatrix} 5 \\ 11 \end{pmatrix}$$

$$\underset{2\times1}{\begin{pmatrix} 1 \\ 2 \end{pmatrix}} \cdot \underset{2\times2}{\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}} \quad \text{not defined}$$

$$2\times1 \neq 2\times2$$

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<u>Notice:</u> $\boxed{A \cdot B \neq B \cdot A}$ for matrices

<u>Ex:</u> Assume A, B are 2×2-matrices

$$(A+B)^2 = \underbrace{(A+B)}_{2\times2} \cdot \underbrace{(A+B)}_{2\times2}$$

$$= A\cdot A + A\cdot B + B\cdot A + B\cdot B$$

$$= A^2 + \underbrace{AB+BA}_{\neq\, 2AB} + B^2$$

<u>Motivation:</u> Linear systems

$$\begin{aligned} x+y+z &= 3 \\ x+2y+4z &= 7 \\ x+7y+9z &= 13 \end{aligned} \quad \xrightarrow[\text{matrix}]{\text{augmented}} \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right)$$

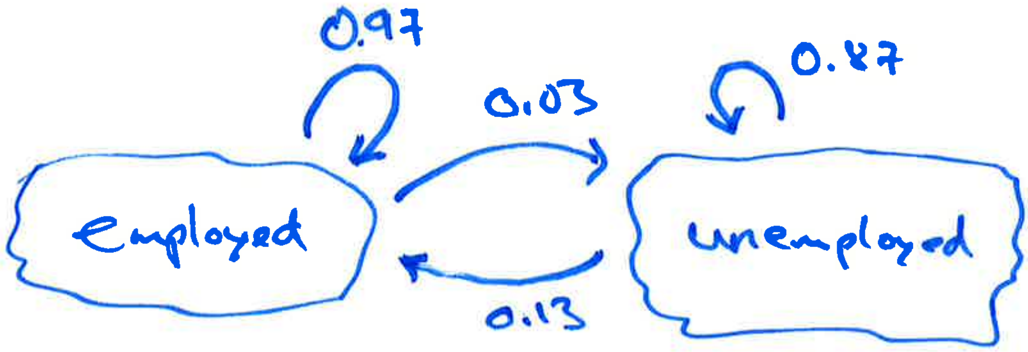
$\Big\{$ coeff. matrix
var. vector, const. vector

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \qquad \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 3 \\ 7 \\ 13 \end{pmatrix}$$

$$A\cdot\underline{x} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1\cdot x + 1\cdot y + 1\cdot z \\ 1\cdot x + 2\cdot y + 4\cdot z \\ 1\cdot x + 3\cdot y + 9\cdot z \end{pmatrix} = \begin{pmatrix} x+y+z \\ x+2y+4z \\ x+3y+9z \end{pmatrix} = \overset{\underline{b}}{\overset{\parallel}{\begin{pmatrix} 3 \\ 7 \\ 13 \end{pmatrix}}}$$

<u>Matrix form:</u> $\boxed{A\cdot\underline{x} = \underline{b}}$

Ex:



State vector: $\underline{x}_t = \begin{pmatrix} e_t \\ u_t \end{pmatrix}$ employed / unemployed rate in week t

$$\underline{x}_t = \begin{pmatrix} e_0 \\ u_0 \end{pmatrix} = \begin{pmatrix} 0.90 \\ 0.10 \end{pmatrix} \qquad A = \begin{pmatrix} 0.97 & 0.13 \\ 0.03 & 0.87 \end{pmatrix}$$

transition matrix

week 0:

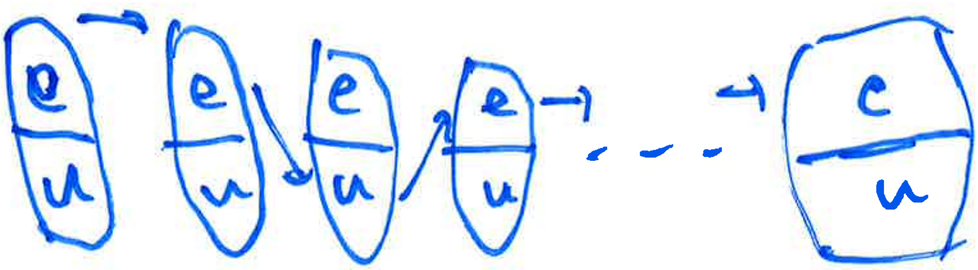
$$\underline{x}_0 = \begin{pmatrix} 0.90 \\ 0.10 \end{pmatrix}$$

week 1:

$$\underline{x}_1 = \begin{pmatrix} 0.97 & 0.13 \\ 0.03 & 0.87 \end{pmatrix} \cdot \begin{pmatrix} 0.90 \\ 0.10 \end{pmatrix} = A \cdot \underline{x}_0$$

$$= \begin{pmatrix} 0.97 \cdot 0.90 + 0.13 \cdot 0.10 \\ 0.03 \cdot 0.90 + 0.87 \cdot 0.10 \end{pmatrix}$$

$$= \begin{pmatrix} e_1 \\ u_1 \end{pmatrix}$$



$$\underline{x}_{100} = A^{100} \cdot \underline{x}_0$$

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Special matrices:

i) **Zero matrix:** $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ (2x2-case)

$A + O = A$

Algebraic rules:

$A + B = B + A$

$(A+B)\cdot C = AC + BC$

$\vdots$

$A + O = A$

ii) **Identity matrix:** $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ (3x3-case)

$A \cdot I = A$

1's on main diagonal
0's elsewhere

Ex: $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}$ $\quad I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$A \cdot I = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} = A$$

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Inverse matrices:

Recall:

division by 2

= multiplication by $1/2 = 2^{-1}$

$\frac{1}{a}$: inverse of a (for $a \neq 0$)

Defn:

Let A be an n×n-matrix
(square matrix: #rows = #cols)

An inverse of A is a matrix B such that

$$A \cdot B = I$$

and

$$B \cdot A = I$$

If it exists, then it is unique, and it is called A^{-1} ($= B$).

$$a \cdot \underbrace{\frac{1}{a}}_{\text{inverse of a}} = 1$$

Ex: $A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$ $\quad A^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$ ok.

$$A \cdot A^{-1} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

$$A^{-1} \cdot A = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$$

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Ex:

$$\begin{aligned} x + 2y &= 7 \\ y &= 13 \end{aligned} \rightsquigarrow A = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \quad \underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} \quad \underline{b} = \begin{pmatrix} 7 \\ 13 \end{pmatrix}$$

$$\downarrow$$

$$A^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$$

Matrix form:

$$A \cdot \underline{x} = \underline{b}$$

$$\underbrace{A^{-1} \cdot A}\underline{x} = A^{-1} \cdot \underline{b}$$

$$I \cdot \underline{x} = A^{-1} \cdot \underline{b}$$

$$\underline{x} = A^{-1} \cdot \underline{b} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 13 \end{pmatrix} = \underline{\underline{\begin{pmatrix} -19 \\ 13 \end{pmatrix}}}$$

$$\underline{\underline{x = -19, \ y = 13}}$$

Ex: $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad A^{-1} = ?$

$$A \cdot A^{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

is not possible

$$\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix}$$

$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ has no inverse

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Method for finding inverses:

A $n \times n$-matrix (square)

$$(A \mid I) \xrightarrow[\text{elementary row operations}]{} \cdots \rightarrow (E \mid B)$$

E: reduced echelon form

Result:

$E = I$: $A^{-1} = B$

$E \neq I$: A^{-1} does not exist

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 0 & 7 & 4 \\ 0 & 0 & 1 \end{pmatrix}$

$$\frac{8}{7} \cdot 3 = \frac{8-21}{7}$$

$$(A \mid I) = \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 7 & 4 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \cdot 1/7$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 3 & 1 & 0 & 0 \\ 0 & 1 & 4/7 & 0 & 1/7 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \begin{matrix} \\ \\ -3,\ -4/7 \end{matrix} \rightarrow \left(\begin{array}{ccc|ccc} 1 & 2 & 0 & 1 & 0 & -3 \\ 0 & 1 & 0 & 0 & 1/7 & -4/7 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) \begin{matrix} \\ -2 \\ \\ \end{matrix}$$

$$\rightarrow \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2/7 & -13/7 \\ 0 & 1 & 0 & 0 & 1/7 & -4/7 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right) = (I \mid A^{-1})$$

reduced echelon form

$$A^{-1} = \begin{pmatrix} 1 & -2/7 & -13/7 \\ 0 & 1/7 & -4/7 \\ 0 & 0 & 1 \end{pmatrix} = \frac{1}{7} \begin{pmatrix} 7 & -2 & -13 \\ 0 & 1 & -4 \\ 0 & 0 & 7 \end{pmatrix}$$

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Extra material:

Why the method (Gauss-Jordan elimination) works for

- determining if the inverse matrix exists
- finding the inverse matrix

Elementary matrices:

For every elementary row operation, there is a matrix E such that left multiplication by E gives the elementary row operation. The matrices E are called elementary matrices.

Ex:

$$A = \begin{pmatrix} 1 & 2 \\ 7 & 4 \end{pmatrix} \begin{matrix} \curvearrowleft \\ \end{matrix} \to \begin{pmatrix} 7 & 4 \\ 1 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 7 & 4 \end{pmatrix} = \begin{pmatrix} 7 & 4 \\ 1 & 2 \end{pmatrix}$$

$E = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the elementary matrix corresponding to switching row 1 and 2.

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \; -1 \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & 3 & 9 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & 3 & 9 \end{pmatrix}$$

$E = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is the elementary matrix corresponding to

$R(2) := R(2) + (-1) \cdot R(1)$

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Method:

A $n \times n$-matrix

$(A \mid I) \to \cdots\cdots \to (E \mid B)$

Gauss-Jordan process corresponding to elementary matrices $E_1 \; E_2 \cdots E_r$

Then: If $E = I$, we have

$A \to E_1 A \to E_2 E_1 A \to \cdots \to \boxed{E_r \cdots E_2 E_1 A = I}$

$I \to E_1 I \to E_2 E_1 I \to \cdots \to E_r \cdots E_2 E_1 \cdot I$

$E_1 I = E_1$, $\quad E_2 E_1 I = E_2 E_1$, $\quad E_r \cdots E_2 E_1 \cdot I = E_r \cdots E_2 E_1$

this means that

$(E_r \cdots E_2 E_1) \cdot A = I$

$= B \quad E_r \cdots E_2 E_1 = A^{-1}$

by defn.