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LECTURE 3

FORK 1003
MATHEMATICS

EIVIND ERIKSEN

AUG 2, 2017

Plan:

1. Determinants
2. Inverse matrices and determinants.
3. Cramer's rule and linear systems

Review:

Matrices and vectors.

Addition / subtraction, scalar multiplication $A+B$, $A-B$, rA ($\underline{v}+\underline{w}$, $\underline{v}-\underline{w}$, $r\underline{v}$)

Matrix multiplication. Identity matrix I.

Inverse matrices: Inverse of A is a matrix A^{-1} such that

$$\boxed{\begin{array}{l} A^{-1}A = I \\ A \cdot A^{-1} = I \end{array}} \leftarrow \text{defn.}$$

Method: $(A \mid I) \xrightarrow{\text{Gauss-Jordan elimination}}$ Reduced echelon form

- $= (I \mid A^{-1})$ — A^{-1} exists and is
- $(\neq I \mid *)$ — A^{-1} does not exist

Linear system:

Matrix form: $A \cdot \underline{x} = \underline{b}$ — assume A^{-1} exists

$$A^{-1}(A\underline{x}) = A^{-1} \cdot \underline{b} \qquad \boxed{\underline{x} = A^{-1} \cdot \underline{b}}$$

One unique solution

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Matrix algebra rules:

$$\left.\begin{aligned} A+B &= B+A \\ A\cdot(B+C) &= A\cdot B + A\cdot C \\ A\cdot(BC) &= (AB)\cdot C \\ &\vdots \end{aligned}\right\}$$

most rules are the same as for numbers

But:

$$A\cdot B \neq B\cdot A$$

no division for matrices ← Must instead multiply with the inverse matrix (if it exists; must choose left or right mult.)

Transpose of a matrix A : A^T, A^t, A^*

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$ $\quad A^T = \begin{pmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{pmatrix}$

$2\times 3 \qquad 3\times 2$

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$$

Defn: A is called symmetric if $A^T = A$

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \qquad A^T = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = A.$$

Rules for transposes:

$$(A+B)^T = A^T + B^T \qquad (r\cdot A)^T = r\cdot A^T$$

$$(A-B)^T = A^T - B^T \qquad \boxed{(A\cdot B)^T = B^T\cdot A^T}$$

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1 Determinants

A $n\times n$-matrix (square) $\rightsquigarrow$ $\det(A) = |A|$ determinant of A (a number)

$\boxed{\begin{array}{l}\underline{n=2 \text{ case:}} \\ A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \quad |A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \underline{ad-bc}\end{array}}$

Ex: $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 1\cdot 4 - 2\cdot 3 = 4-6 = \underline{\underline{-2}}$

Application: Inverse matrices

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{cases} A^{-1} = \dfrac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}, \quad \overset{\overset{|A|}{\shortparallel}}{ad-bc} \neq 0 \\ A^{-1} \text{ does not exist if } \overset{\overset{|A|}{\shortparallel}}{ad-bc} = 0 \end{cases}$$

This means: A^{-1} exists $\iff |A| \neq 0$
(A invertible)

Also: $A\underline{x} = \underline{b}$
linear system
(A $n\times n$-matrix)

$|A| \neq 0$: the linear system has one solution

$|A| = 0$: the linear system has either no solutions or inf. many solutions

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Case $n > 2$: Defn: Formula with $n!$ terms of degree n

$n=3$: $A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix}$

$$|A| = aei + bfg + cdh - ceg - fha - ibd$$

n! terms of degree n

$$= a(ei - fh) + b(fg - id) + c(dh - eg)$$

Methods:

$$= a(ei - fh) - b(id - fg) + c(dh - eg)$$

(a) Cofactor expansion along a row or a column

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$ $\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$

$$|A| = +1 \cdot \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

sign number from matrix minor

$$= 1 \cdot 6 + 1 \cdot (-5) + 1 \cdot 1$$

$$= 6 - 5 + 1 = \underline{\underline{2}}$$

$$= a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + a_{13} \cdot C_{13}$$

cofactors:

$$C_{11} = + \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} = +(2 \cdot 9 - 3 \cdot 4) = \underline{6}$$

$$C_{12} = - \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} = -(1 \cdot 9 - 1 \cdot 4) = \underline{-5}$$

$$C_{13} = + \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} = +(1 \cdot 3 - 1 \cdot 2) = \underline{1}$$

$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$

$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$

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$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = \underbrace{1 \cdot \left(+ \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} \right)}_{a_{11} \cdot C_{11}} + \underbrace{1 \cdot \left(- \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} \right)}_{a_{12} \cdot C_{12}} + \underbrace{1 \cdot \left(+ \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix} \right)}_{a_{13} \cdot C_{13}}$$

$$= +1 \cdot \begin{vmatrix} 2 & 4 \\ 3 & 9 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 2 \\ 1 & 3 \end{vmatrix}$$

$$= 1 \cdot 6 \underbrace{-1 \cdot 5}_{C_{12}} + 1 \cdot 1 = 6 - 5 + 1 = \underline{\underline{2}}$$

$$\boxed{\text{Fact: Cofactor expansion along any row or column will give the same result, } |A|.}$$

Ex: Cofactor expansion along second col.

$$|A| = a_{12} \cdot C_{12} + a_{22} \cdot C_{22} + a_{32} \cdot C_{32}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{vmatrix} = 1 \cdot \left(- \begin{vmatrix} 1 & 4 \\ 1 & 9 \end{vmatrix} \right) + 2 \cdot \left(+ \begin{vmatrix} 1 & 1 \\ 1 & 9 \end{vmatrix} \right) + 3 \left(- \begin{vmatrix} 1 & 1 \\ 1 & 4 \end{vmatrix} \right)$$

$$= -1 \cdot (9-4) + 2(9-1) - 3(4-1) = -5 + 16 - 9 = \underline{\underline{2}}$$

Cofactor expansion works for any n:

$$|A| = a_{11} \cdot C_{11} + a_{12} \cdot C_{12} + a_{13} \cdot C_{13} + \dots + a_{1n} \cdot C_{1n}$$

(along first row in $n \times n$-case)

Cofactors in general:

$$C_{ij} = \underbrace{(-1)^{i+j}}_{\text{sign}} \cdot M_{ij}$$

(row i, col. j)

M_{ij}: minor in position (i,j) = determinant of the submatrix you obtain by deleting row i and col. j.

$$\begin{pmatrix} + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ + & - & + & - & + & \cdots \\ - & + & - & + & - & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

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Ex:

$$\begin{vmatrix} 0 & 7 & 3 & 4 \\ 1 & 0 & -1 & 2 \\ 2 & 1 & 0 & 1 \\ 0 & 4 & -1 & 2 \end{vmatrix} = \cancel{0 \cdot C_{11}} + 1 \cdot C_{21} + 2 \cdot C_{31} + \cancel{0 \cdot C_{41}}$$

$$= 1 \cdot \left(- \begin{vmatrix} 7 & 3 & 4 \\ 1 & 0 & 1 \\ 4 & -1 & 2 \end{vmatrix} \right) + 2 \cdot \left(+ \begin{vmatrix} 7 & 3 & 4 \\ 0 & -1 & 2 \\ 4 & -1 & 2 \end{vmatrix} \right)$$

$$= -1 \cdot \left(1 \cdot \left(- \begin{vmatrix} 3 & 4 \\ -1 & 2 \end{vmatrix} \right) + 1 \cdot \left(- \begin{vmatrix} 7 & 3 \\ 4 & -1 \end{vmatrix} \right) \right)$$

$$+ 2 \cdot \left((-1) \cdot \left(+ \begin{vmatrix} 7 & 4 \\ 4 & 2 \end{vmatrix} \right) + 2 \cdot \left(- \begin{vmatrix} 7 & 3 \\ 4 & -1 \end{vmatrix} \right) \right)$$

$$= -1 \left(-10 - (-19) \right) + 2 \left(-(-2) - 2(-19) \right)$$

$$= -9 + 80 = \underline{\underline{71}}$$

Alternative method:

Ex:

$$\left| \begin{matrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{matrix} \right| \begin{matrix} 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{matrix} = 18 + 4 + 3 - 2 - 12 - 9$$

$$= 25 - 23 = \underline{\underline{2}}$$

works only for $n=3$

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Determinants using Gaussian elimination:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \longrightarrow B = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 0 & 2 & 8 \end{pmatrix}$$

$$|A| = ?\ |B| = 2 \qquad |B| = +1 \cdot \begin{vmatrix} 1 & 3 \\ 2 & 8 \end{vmatrix} = 2$$

Fact:

If you do one elementary row operation $A \to B$, then we have:

- If you switch two rows, then $|B| = -|A|$
- If you multiply a row by $c \neq 0$, then $|B| = c \cdot |A|$
- If you add a multiple of one row to another row, then $|B| = |A|$

Ex: $|A| = |B| = 71$ $\qquad |B| = (-1)\cdot(-1)\cdot|A|$

$$A = \begin{pmatrix} 0 & 7 & 3 & 4 \\ 1 & 0 & -1 & 2 \\ 2 & 1 & 0 & 1 \\ 0 & 4 & -1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 7 & 3 & 4 \\ 2 & 1 & 0 & 1 \\ 0 & 4 & -1 & 2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 7 & 3 & 4 \\ 0 & 1 & 2 & -3 \\ 0 & 4 & -1 & 2 \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -3 \\ 0 & 7 & 3 & 4 \\ 0 & 4 & -1 & 2 \end{pmatrix}$$

$$\longrightarrow \begin{pmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 2 & -3 \\ 0 & 0 & -11 & 25 \\ 0 & 0 & -9 & 14 \end{pmatrix} = B$$

$$|B| = 1 \cdot \begin{vmatrix} 1 & 2 & -3 \\ 0 & -11 & 25 \\ 0 & -9 & 14 \end{vmatrix}$$

$$= 1 \cdot 1 \cdot (-11 \cdot 14 + 9 \cdot 25)$$

$$= 225 - 154 = 71$$

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<u>Fact</u>: If A is a upper triangular square matrix:

$$A = \begin{pmatrix} a_1 & * & * & \cdots & * \\ 0 & a_2 & * & \cdots & * \\ 0 & 0 & a_3 & \cdots & * \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_n \end{pmatrix}$$

all entries under the main diagonal are <u>zero</u>

then $\underline{|A| = a_1 \cdot a_2 \cdot \dots \cdot a_n}$

$$\begin{vmatrix} a_1 & * & * & * \\ 0 & a_2 & * & * \\ 0 & 0 & a_3 & * \\ 0 & 0 & 0 & a_4 \end{vmatrix} = a_1 \begin{vmatrix} a_2 & * & * \\ 0 & a_3 & * \\ 0 & 0 & a_4 \end{vmatrix}$$

$$= a_1 \cdot \left(a_2 \cdot \begin{vmatrix} a_3 & * \\ 0 & a_4 \end{vmatrix} \right) = a_1 \cdot a_2 \cdot (a_3 a_4)$$

$$= \underline{a_1 \cdot a_2 \cdot a_3 \cdot a_4}$$

(2) Determinants and inverses:

A $n \times n$-matrix:

$\boxed{|A| \neq 0 \Leftrightarrow A^{-1} \text{ exists (A is invertible)}}$

Formula for A^{-1}:

If $|A| \neq 0$, then

$$\boxed{A^{-1} = \frac{1}{|A|} \cdot \operatorname{adj}(A)}$$

$$\operatorname{adj}(A) = \begin{pmatrix} C_{11} & C_{12} & C_{13} & \cdots & C_{1n} \\ C_{21} & C_{22} & C_{23} & \cdots & C_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ C_{n1} & C_{n2} & C_{n3} & \cdots & C_{nn} \end{pmatrix}^T$$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$;

$ad - bc \neq 0$:

$$A^{-1} = \frac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Ex:

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$$

$$|A| = 1 \cdot 6 + 1 \cdot (-5) + 1 \cdot 1 = \underline{2}$$

$C_{11} = 6 \quad C_{12} = -5 \quad C_{13} = 1$

extra comp. of cofactors $\begin{cases} C_{21} = -6 \quad C_{22} = 8 \quad C_{23} = -2 \\ C_{31} = 2 \quad C_{32} = -3 \quad C_{33} = 1 \end{cases}$

$$A^{-1} = \underset{|A|}{\frac{1}{2}} \cdot \underbrace{\begin{pmatrix} 6 & -5 & 1 \\ -6 & 8 & -2 \\ 2 & -3 & 1 \end{pmatrix}^T}_{\operatorname{adj}(A)} = \underline{\underline{\frac{1}{2} \begin{pmatrix} 6 & -6 & 2 \\ -5 & 8 & -3 \\ 1 & -2 & 1 \end{pmatrix}}}$$

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3 Cramer's rule

Linear system $A\underline{x} = b$ such that $\begin{cases} \text{i) } A \text{ is } n\times n \text{ (square)} \\ \text{ii) } |A| \neq 0 \end{cases}$

In this case, we know that there is one unique solution

$$A\underline{x} = \underline{b}$$
$$A^{-1}A\underline{x} = A^{-1}\underline{b}$$
$$\underline{x} = A^{-1}\underline{b}$$

since $|A| \neq 0 \Rightarrow A^{-1}$ exists.

Cramer's rule: Formula for this solution (that is, another formula)

$$\underline{x_1} = \frac{|A_1(\underline{b})|}{|A|} \longleftarrow A_1(\underline{b}) \text{: replace col. 1 in } A \text{ with } \underline{b}$$

$$x_2 = \frac{|A_2(\underline{b})|}{|A|} \longleftarrow A_2(\underline{b}) \text{: replace col. 2 in } A \text{ with } \underline{b}$$

$$\vdots$$

$$x_n = \frac{|A_n(\underline{b})|}{|A|} \longleftarrow A_n(\underline{b}) \text{: replace col. } n \text{ in } A \text{ with } \underline{b}$$

Ex:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3 \\ 7 \\ 13 \end{pmatrix}$$

$|A| = 2 \neq 0 \Rightarrow$ one solution (x, y, z)

$$\left.|A_1(b)| = \begin{vmatrix} 3 & 1 & 1 \\ 7 & 2 & 4 \\ 13 & 3 & 9 \end{vmatrix} \begin{aligned} &= 3\cdot 6 - 1\cdot 11 + 1\cdot(-5) \\ &= 18 - 11 - 5 = \underline{2} \end{aligned}\right\} \Rightarrow x = \frac{|A_1(b)|}{|A|} = \frac{2}{2}$$

$$\underline{\underline{x = 1}}$$

(similar for y and z)