BI HANDELSHØYSKOLEN
Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo
Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500
www.bi.no
info@bi.no
Org. nr. 971 22 8865

LECTURE 4

FORK 1003 MATHEMATICS

EIVIND ERIKSEN, AUG 3, 2017

Plan:

1. Functions and the derivative
2. Computing the derivative
3. Exponential functions and logarithms
4. Higher order derivatives
5. Applications

(4 and 5: postponed to Lecture 5)

Review:

Determinants (A $n\times n$-matrix):
- computation using cofactor expansion
- " " " Gaussian elimination

Fact: A invertible (A^{-1} exists) $\Leftrightarrow$ $|A|\neq 0$

If $|A|\neq 0$, then $A^{-1} = \frac{1}{|A|}\cdot \operatorname{adj}(A) = \frac{1}{|A|}\cdot \begin{pmatrix} C_{11} & C_{12} & \cdots & C_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ C_{n1} & C_{n2} & \cdots & C_{nn} \end{pmatrix}^T$

n=2:

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $|A| = ad-bc$

If $ad-bc\neq 0$, then $A^{-1} = \frac{1}{ad-bc}\cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

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Rules for determinants:

i) $\boxed{\det(A\cdot B) = \det(A)\cdot\det(B)}$

Ex: If $\det(A)=3$, then $\det(A^2) = \det(A)\cdot\det(A) = \underline{9}$

ii) $\det(A^{-1}) = \dfrac{1}{\det(A)}$

$A^{-1}\cdot A = I$

$|A^{-1}\cdot A| = |I| = \begin{vmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{vmatrix}$

$|A^{-1}|\cdot|A| = 1$

$|A^{-1}| = 1/|A|$

iii) $\det(r\cdot A) = r^n\cdot\det(A)$

iv) $\det(A^T) = \det(A)$

Ex: $A = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = 2\cdot\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$|A| = 4 = 2^2\cdot 1$

Rules for inverses:

i) $\boxed{(A\cdot B)^{-1} = B^{-1}\cdot A^{-1}}$

$(B^{-1}A^{-1})\cdot(A\cdot B)$

$= B^{-1}\cdot \underbrace{A^{-1}\cdot A}\cdot B \neq A^{-1}B^{-1}AB$

$= B^{-1}\cdot I\cdot B$

$B^{-1}\cdot B = I$

ii) $(A^T)^{-1} = (A^{-1})^T$

$(A^{-1})^T\cdot A^T = (A\cdot A^{-1})^T = I^T = I$

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Ex:

$$\begin{vmatrix} 1 & 0 & 0 & -1 & 0 \\ 4 & 7 & 1 & 2 & 1 \\ 1 & 1 & -1 & 0 & 1 \\ 3 & 2 & 1 & 1 & -1 \\ 0 & 1 & 4 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 1 & 3 & 0 \\ 0 & 7 & 1 & 2 & 1 \\ 0 & 1 & -1 & 1 & 4 \\ -1 & 2 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 & 2 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 4 & 1 & 3 & 0 \\ 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & \cdot & \cdot & \cdot & \cdot \\ 0 & 6 & 1 & 4 & 1 \\ 0 & \cdot & \cdot & \cdot & \cdot \end{vmatrix} \leftarrow \text{one } 4\times4 \text{ determinant}$$

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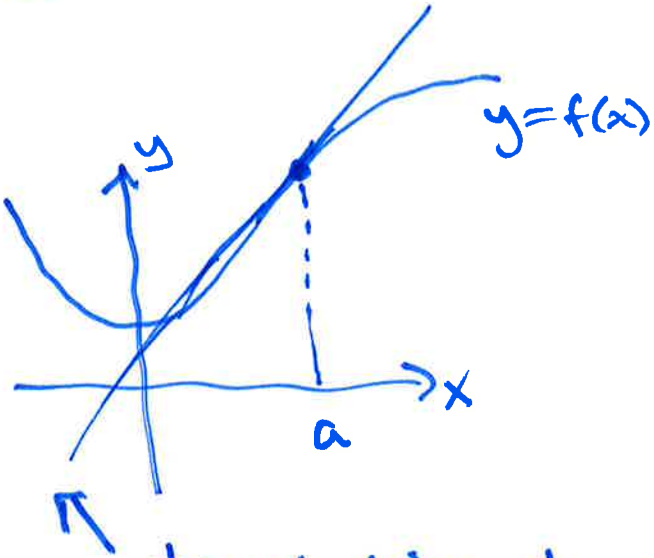
1 Functions and derivative in one variable

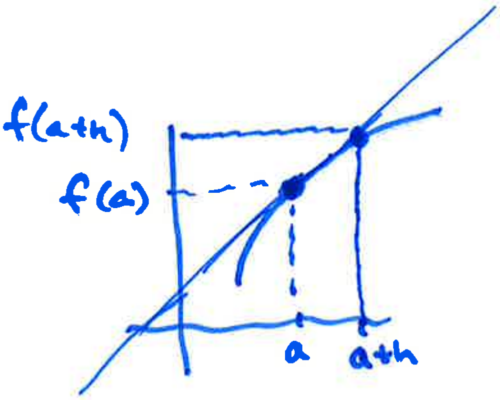
Ex: $f(x) = x^2 + 1$



The graph of a function in one variable.

tangent line of $y=f(x)$ at $x=a$

Slope of tangent line: $f'(a)$



line with slope $\frac{\Delta y}{\Delta x} = \frac{f(a+h)-f(a)}{h}$

called a secant line

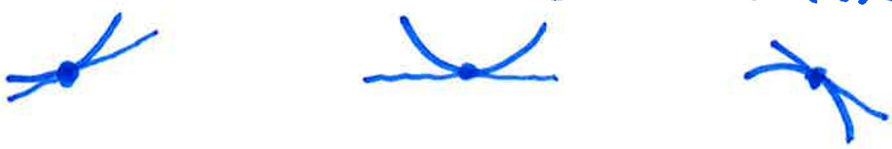
when h is small, the secant line is a good approximation of the tangent line.

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}}$$

The derivative of f at $x=a$.

Interpretation:

$f'(a) > 0$ $\quad$ $f'(a) = 0$ $\quad$ $f'(a) < 0$



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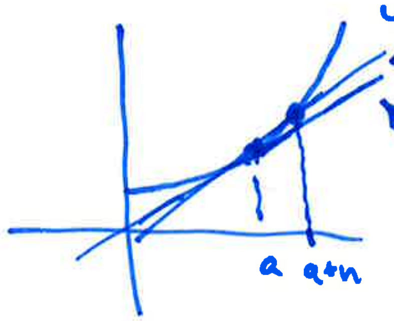
$$f'(a) \simeq \frac{f(a+h)-f(a)}{h} \simeq \frac{f(a+1)-f(a)}{1} = f(a+1)-f(a)$$

(small h) $\qquad$ ($h=1$)



secant through the two pts

tangent

② Computing the derivative

The derivative as a function: $f'(x) = \lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

Ex: $f(x)=x^2$

$f'(x) = \underline{2x}$

Power rule:

$$\boxed{(x^n)' = n \cdot x^{n-1}}$$

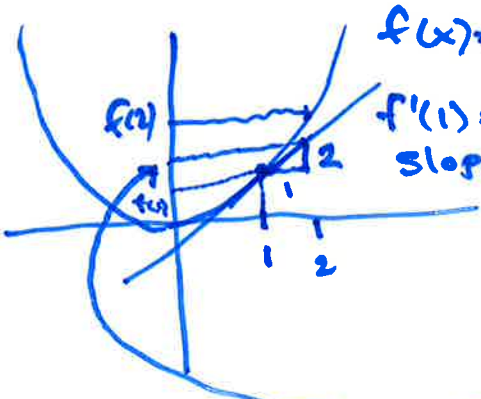
By the defn:

$$f'(x) = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h}$$

$$= \lim_{h\to 0} (2x+h) = \underline{\underline{2x}}$$

$f'(1) = 2\cdot 1 = \underline{2}.$



$f(x)=x^2$

$f'(1)=2$

slope of tangent

$$f(2) \simeq f(1) + 1\cdot f'(2)$$

$$f(2)-f(1) \simeq 1\cdot f'(2)$$

$$\frac{f(2)-f(1)}{1} \simeq f'(1)$$

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Derivation rules:

① Power rule: $(x^n)' = n \cdot x^{n-1}$ (any n)

② Add./subtr: $(u+v)' = u' + v'$

$(u-v)' = u' - v'$

③ Scalar mult: $(r \cdot u)' = r \cdot u'$ (r number, $u = u(x)$ expr.)

④ Product: $(u \cdot v)' = u'v + u \cdot v'$

⑤ Quotient: $\left(\frac{u}{v}\right)' = \frac{u' \cdot v - u v'}{v^2}$

Ex:

$$\left(x^2 + 1 - \frac{1}{x}\right)' = (x^2)' + (\overset{x^0}{1})' - (\overset{x^{-1}}{\tfrac{1}{x}})'$$

$$= 2x + 0 - (-1) \cdot x^{-2}$$

$$= \underline{2x + \frac{1}{x^2}}$$

Notation:

$f(x) = 1/x \longrightarrow \overset{f'}{\overset{\shortparallel}{f'(x)}} = -1/x^2$

$(1/x)' = -1/x^2$

$\frac{df}{dx}(x) = \frac{df}{dx} = -\frac{1}{x^2}$

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6 Chain rule:

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

understood that u is the kernel in a composite fn.

$$f'(x) = f'(u) \cdot u'(x)$$

Ex: $\left(\frac{1}{x^2+1}\right)' = \left[(x^2+1)^{-1}\right]'$

$$f(x) = \frac{1}{x^2+1}$$

$$\shortparallel$$

$$f(u) = \frac{1}{u} \quad \text{with} \quad u = x^2+1$$

$$\begin{tikzcd}
x \arrow[rr, "\text{composite fn.}"] \arrow[dr] & & \frac{1}{x^2+1} \\
& u = x^2+1 \arrow[ur] &
\end{tikzcd}$$

$u(x) = x^2+1$ (inner function, kernel)

$u \mapsto \frac{1}{u}$ (outer function)

$$\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$$

$$\begin{aligned}
f'(x) &= f'(u) \cdot u'(x) \\
&= \left(\tfrac{1}{u}\right)'_u \cdot (x^2+1)'_x \\
&= \frac{d}{du}\left(\overset{u^{-1}}{1/u}\right) \cdot \frac{d}{dx}(x^2+1) \\
&= -\frac{1}{u^2} \cdot 2x = -\frac{1}{(x^2+1)^2} \cdot 2x = \underline{\underline{-\frac{2x}{(x^2+1)^2}}}
\end{aligned}$$

$$\frac{df}{dx} = \frac{df}{du}\cdot\frac{du}{dx}$$

Ex: $f(x) = \sqrt{x^2+4} = \sqrt{u} = u^{1/2}$ with $u = x^2+4$

$$f'(x) = f'(u)\cdot u'(x) = \left(\frac{1}{2}u^{-1/2}\right)\cdot 2x$$

$$= \frac{\cancel{2}x}{\cancel{2}\,u^{1/2}} = \frac{x}{\sqrt{u}} = \underline{\underline{\frac{x}{\sqrt{x^2+4}}}}$$

$g(x) = (2-x)^3 = u^3$ with $u = 2-x$

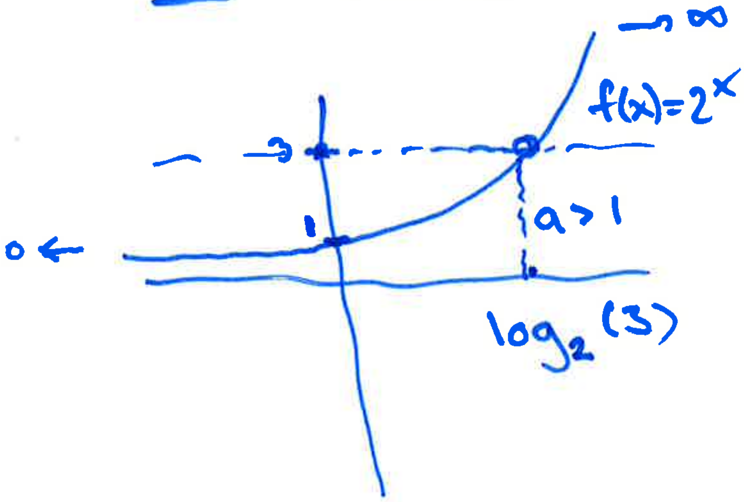
$$g'(x) = 3u^2\cdot(-1) = -3u^2 = \underline{\underline{-3(2-x)^2}}$$

3 Exponential functions and logarithms

i) Exponential functions:

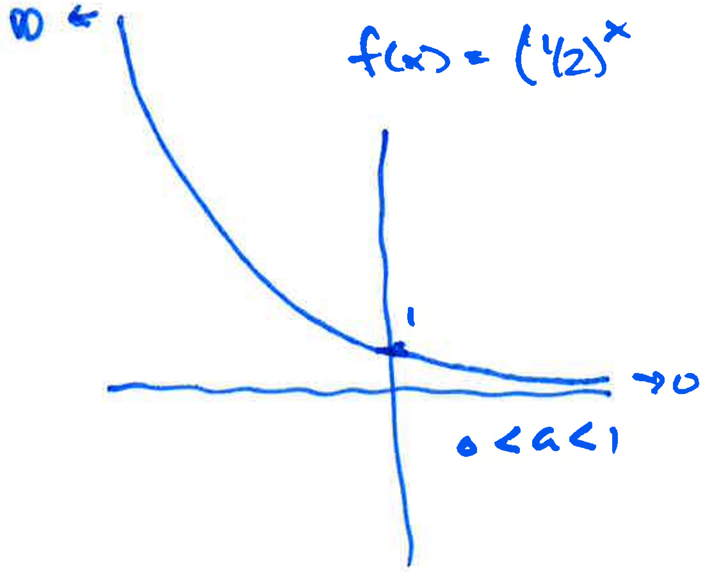
$f(x) = a^x$ (with $a>0$)

Ex: $f(x) = 2^x$



$(2^{-1})^x = 2^{-x}$

$f(x) = (1/2)^x$



Derivation rule:

$$(a^x)' = a^x \cdot \ln(a) \longleftarrow$$ $\ln(a)$: natural logarithm.

ii) Logarithms:

Recall:

$$\boxed{\begin{aligned} a^x \cdot a^y &= a^{x+y} \\ a^x : a^y &= a^{x-y} \\ (a^x)^y &= a^{x \cdot y} \end{aligned}}$$

Defn: logarithm with base a $(a>0)$

$\log_a(x)$ = the unique number y s.t.

$$\boxed{a^y = x}$$

$f(x) := \log_a(x)$: logarithm with base a defined for $x>0$

It is unique if it exists.

$\log_a(x)$ is the inverse function of a^x

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Rules for logarithms:

i) $\log_a(x \cdot y) = \log_a(x) + \log_a(y)$

ii) $\log_a(x/y) = \log_a(x) - \log_a(y)$

iii) $\log_a(\ \ ) = y \cdot \log_a(x)$

$\uparrow$ x^y

Euler number and natural logarithms

Euler number: $e = 2{,}71828\ldots \simeq 2{,}72$

$f(x) = \ln(x)$: natural logarithm $= \log_e(x)$

($f(x) = \log(x)$: $\log_{10}(x)$)
historical

$$\boxed{\begin{array}{l}\text{Rule:}\\ \text{iv) } \log_a(x) = \ln(x)/\ln(a)\end{array}}$$

Important property of e:

$a = e$ is the unique number a s.t. $(a^x)' = a^x$

Restatement of derivation rules for exp. fns:

i) $\boxed{(e^x)' = e^x}$

ii) $(a^x)' = a^x \cdot \ln(a)$

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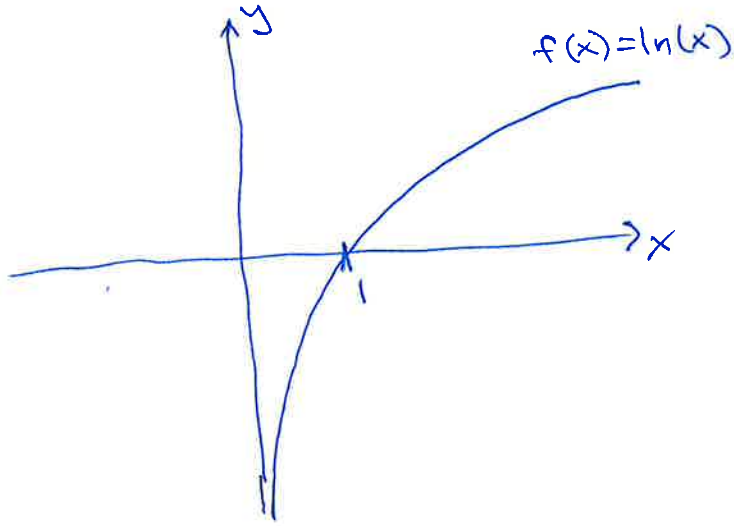
Derivation rules for logarithms:

i) $\boxed{(\ln x)' = \frac{1}{x}}$

ii) $(\log_a(x))' = \frac{1}{x} \cdot \frac{1}{\ln(a)}$

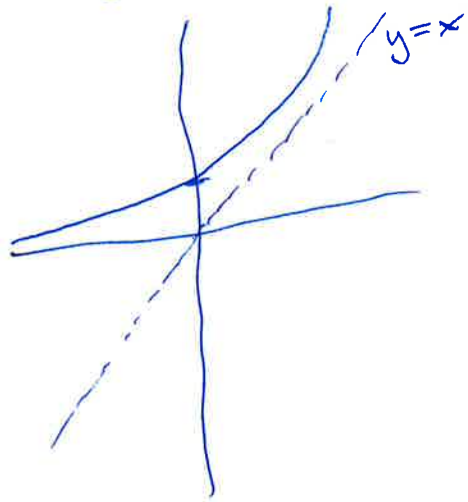
$(a = e \approx 2{,}72)$

Graphs of logarithms:



(defined for $x > 0$)

Compare with the graph of $f(x) = e^x$, the inverse fn:



mirror image along $y = x$