LECTURE 5

FORK 1003

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MATHEMATICS

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B4-032

Plan:

1. Higher order derivatives
2. Optimization in one variable
3. Integration

Review:

Differentiation: fn. in one variable

- computing derivatives using derivation rules
- interpret the derivative
- exponentials/logarithms: e^x, $\ln x$
- chain rule: $\frac{df}{dx} = \frac{df}{du} \cdot \frac{du}{dx}$

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1 Higher order derivatives

Ex: $f(x) = 3e^{-x}$ $\quad e^u,\ u=-x \quad \frac{df}{du} \quad \frac{du}{dx}$

$f'(x) = 3\cdot(e^{-x})' = 3\cdot e^{-x}\cdot(-1)$

$= -3e^{-x}$

$f''(x) = (-3e^{-x})' = -3\cdot e^{-x}\cdot(-1) = 3e^{-x}$

$f'''(x) = (3e^{-x})' = 3\cdot e^{-x}\cdot(-1) = -3e^{-x}$

$\vdots$

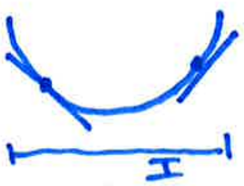
Defn: $f^{(n)}(x) :=$ the function you obtain after differentiating $f(x)$ n times.

Ex: $f(x) = x^3$

$f'(x) = 3x^2$

$f''(x) = 3\cdot 2x = 6x$

Interpretation: of $f''(x)$

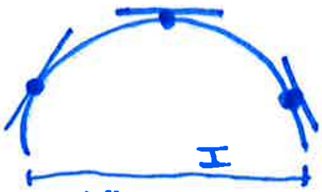


I

$f''(x) > 0$

the slope of the tangent line increases (as x increases)

$\boxed{f \text{ is convex in } I \text{ if } f''(x) \geq 0 \text{ for all } x \text{ in } I}$



I

$f''(x) < 0$

the slope of the tangent line is decreasing (as x increases)

$\boxed{f \text{ is concave in } I \text{ if } f''(x) \leq 0 \text{ for all } x \text{ in } I}$

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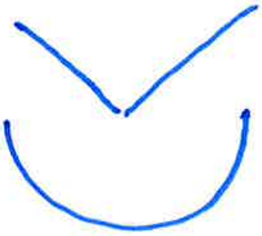
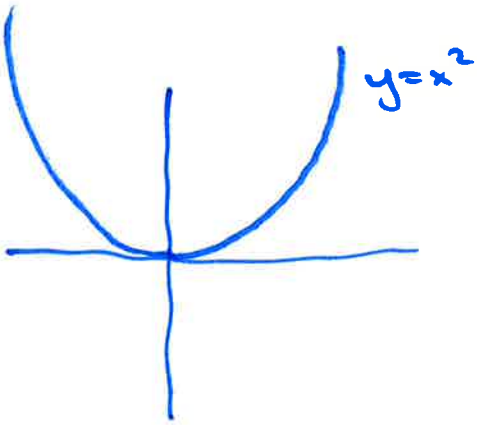
Ex: $f(x) = x^2$

$f'(x) = 2x$

$f''(x) = 2$

f is convex in $\mathbb{R} = (-\infty, \infty)$



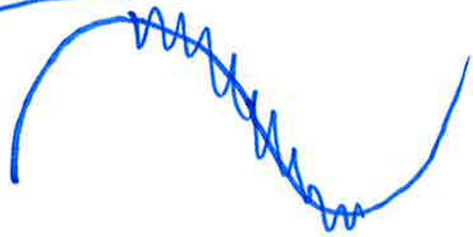


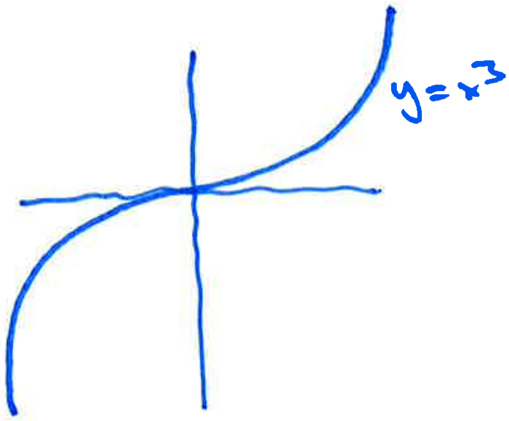
$f(x) = x^3$

$f'(x) = 3x^2$

$f''(x) = 6x$

$f''(x) \leq 0$ when $x \leq 0$	$f''(x) \geq 0$ when $x \geq 0$
concave in $(-\infty, 0]$	convex in $[0, \infty)$





Increasing / decreasing fn:

f is increasing in I if $f'(x) \geq 0$ for all x in I	f is decreasing in I if $f'(x) \leq 0$ for all x in I

$f(x) = x^2$
$f'(x) = 2x$

f decreasing for $x \leq 0$

f increasing for $x \geq 0$

$f(x) = x^3$
$f'(x) = 3x^2 \geq 0$ for all x

f increasing

② Optimization problems

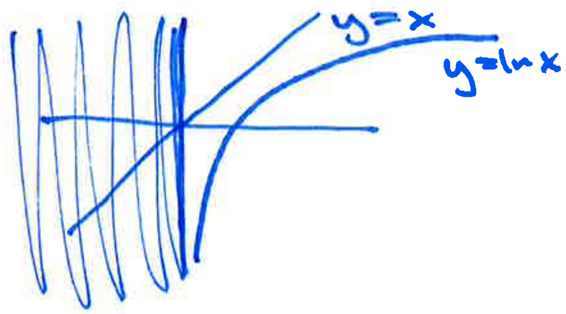
max/min - problems in one variable

$$\boxed{\max/\min \ f(x)}$$

Ex: $f(x) = x \cdot \ln x \ , \ x > 0$

$\max f(x) = \infty$

(has no max)

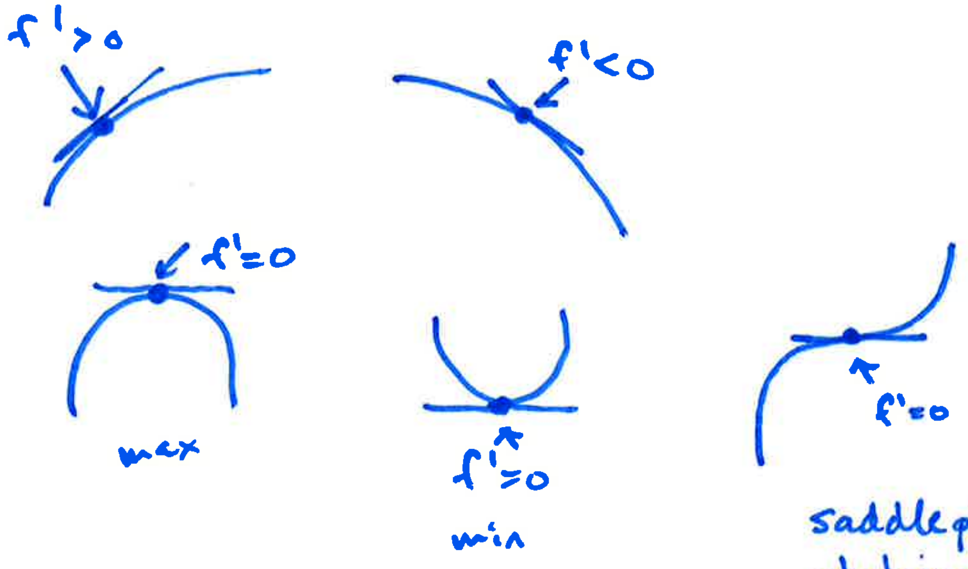


$\min f(x)$ does not exist

Result: $\boxed{x \text{ is max/min for } f \implies f'(x) = 0}$

Stationary pts for f:

$x : f'(x) = 0$



saddle pt. = stationary pt. that is neither local max/min

stationary pts.

local max: x^* s.t. $f(x^*) \geq f(x)$ for x close to x^*

local min: x^* " $f(x^*) \leq f(x)$ — " —

saddle pt: stationary pt that is neither local max nor local min

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$f'(x) = 0$ first order condition (FOC)

Ex: $f(x) = x \cdot \ln x$

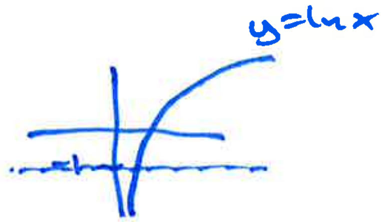
Stad. pts: $f'(x) = 1 \cdot \ln x + x \cdot \frac{1}{x} = \ln x + 1 = 0$

$\ln x = -1$

$e^{\ln x} = e^{-1}$

Stationary pts of f, $\longrightarrow$ $\underline{x = 1/e}$

= candidate for max/min.



Sign of $f'(x) = \ln x + 1$

	0		1/e	
$\ln x + 1$		$-$	0	$+$

$x = 1/e$ is a min. pt. for f

no max

Min value:

$f(1/e) = \frac{1}{e} \cdot (-1)$

$= -1/e$

Method:

- find stationary pts by solving FOC $f'(x) = 0$

  list of candidates for max/min

- make a sign diagram for $f'(x)$

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Ex: $f(x) = x^3 - 2x + 1$

$f'(x) = 3x^2 - 2$

$f''(x) = 6x$

Stat. pts: $f'(x) = 3x^2 - 2 = 0$

$3x^2 = 2$

$x^2 = 2/3$

$x = \pm\sqrt{2/3}$

$x_1 = \sqrt{2/3} \qquad x_2 = -\sqrt{2/3}$

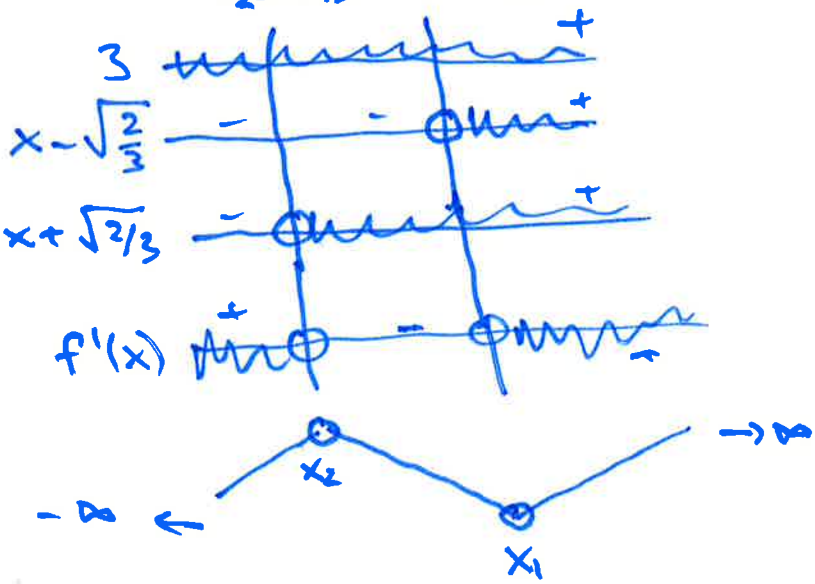
Alt 2: **Classify:**

x_1: $f''(x_1) = 6x_1 = 6\cdot\sqrt{2/3} > 0 \Rightarrow x_1$ local min

x_2: $f''(x_2) = 6x_2 = 6\cdot(-\sqrt{2/3})$
$= -6\sqrt{2/3} < 0 \Rightarrow x_2$ local max

Alt 1: $f'(x) = 3x^2 - 2 = 3(x - \sqrt{2/3})(x + \sqrt{2/3})$

$x_2 = -\sqrt{2/3} \quad x_1 = \sqrt{2/3}$

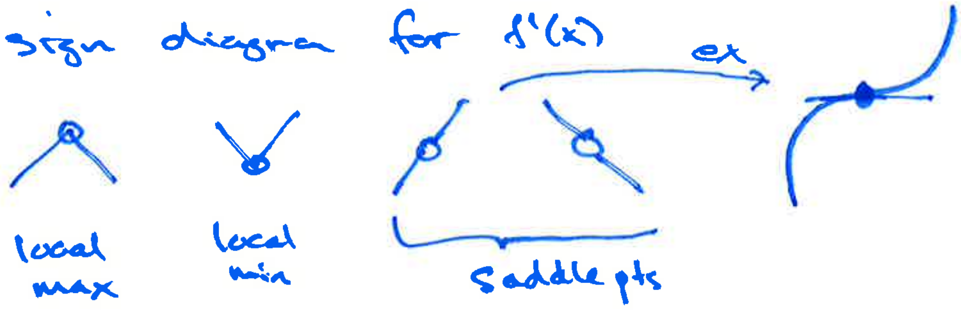


no global max
no global min

General method:

1. Solve FOC: $f'(x) = 0$ to find stationary pts.
2. Classify stationary pts as local max, local min or saddle pts.

Alt I: sign diagram for $f'(x)$



Alt 2:

Second derivative test

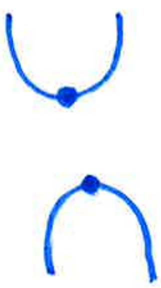
If x^* is a stationary pt, then

$f''(x^*) < 0 \Rightarrow x^*$ local max

$f''(x^*) > 0 \Rightarrow x^*$ local min

no concl. if $f''(x^*) = 0$

3. If there are local max/min, determine if they are global max/min



If f is convex everywhere, then any local min is a global min.

If f is concave everywhere, then any local max is a global max

max = global max min = global min

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3 Integration

int. sign

x is the int. var.

Indefinite integral:
$$\int f(x)\,dx = F(x) + C$$

integrand = the function we want to integrate

We are to find all functions $F(x)$ s.t. $F'(x) = f(x)$.

Ex:
$$\int 2x\,dx = x^2 + C$$
integration constant

$$\int x^3 + 2x - 1\,dx = \frac{1}{4}\cdot x^4 + x^2 - x + C$$

Integration rules:

1. $\int x^n\,dx = \frac{1}{n+1}x^{n+1} + C$, $n \neq -1$
2. $\int \frac{1}{x}\,dx = \ln|x| + C$
3. $\int u+v\,dx = \int u\,dx + \int v\,dx$

   $\int u-v\,dx = \int u\,dx - \int v\,dx$

   $\int r\cdot u(x)\,dx = r\cdot\int u(x)\,dx$ (r constant)
4. $\int e^x\,dx = e^x + C$

   $\int a^x\,dx = a^x\cdot\frac{1}{\ln(a)} + C$ (a>0)

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$$\boxed{\int x^n dx = \frac{1}{n+1} \cdot x^{n+1} + C}$$

$n = 1/2$

Ex: $\int \sqrt{x}\, dx = \int x^{1/2} dx$

$$= \frac{2}{3} \cdot x^{3/2} + C \leftarrow \left(x^{\frac{3}{2}}\right)' = \frac{3}{2} \cdot x^{\frac{1}{2}}$$

$$= \frac{1}{\frac{1}{2}+1} \cdot x^{\frac{1}{2}+1} + C = \frac{1}{3/2} x^{3/2} + C$$

$$= 2/3 \cdot x^{3/2} + C$$

More advanced integration techniques:

i) Integration by parts: (product rule backwards)

$$\boxed{\int u' v \, dx = uv - \int u v' dx}$$

Ex: $\int x e^x dx = \frac{1}{2}x^2 e^x - \int \frac{1}{2}x^2 e^x dx$

$$\cancel{\boxed{\begin{array}{ll} u = \frac{1}{2}x^2 & v = e^x \\ u' = x & v' = e^x \end{array}}}$$

$$\int x e^x dx = x e^x - \int e^x \cdot 1 \, dx$$

$$\boxed{\begin{array}{ll} u = e^x & v = x \\ u' = e^x & v' = 1 \end{array}}$$

$$= x e^x - \int e^x dx = x e^x - e^x + C$$

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$$\int \ln x \, dx = \int 1 \cdot \ln x \, dx =$$

$$\boxed{\begin{matrix} u = x & v = \ln x \\ u' = 1 & v' = 1/x \end{matrix}}$$

$$= x \cdot \ln x - \int \cancel{x} \cdot \frac{1}{\cancel{x}} \, dx = x \ln x - \int 1 \, dx$$

$$= \underline{x \ln x - x + C}$$

ii) <u>Substitution:</u> $\boxed{\begin{matrix} u = u(x) \\ du = u'(x)\,dx \end{matrix}}$ → Use this to transform the int. to one in <u>u</u>. (chain rule backwards)

<u>Ex:</u> $\int x \ln(x^2+1) \, dx = \int \cancel{x} \cdot \ln(u) \cdot \frac{du}{2\cancel{x}}$

$$\boxed{\begin{matrix} u = x^2+1 \\ du = u' \, dx \end{matrix}}$$

$$du = 2x \cdot dx$$

$$\Updownarrow$$

$$dx = \frac{du}{2x}$$

$$= \int \ln(u) \cdot \frac{1}{2} \, du = \frac{1}{2} \int \ln(u) \, du$$

$$= \frac{1}{2} \cdot \left(u \ln u - u \right) + C$$

$$= \underline{\underline{\frac{1}{2}(x^2+1)\ln(x^2+1) - \frac{1}{2}(x^2+1) + C}}$$

Comments about $\int \frac{1}{x}dx$:

We write $\int \frac{1}{x}dx = \ln|x| + C$ with $|x|$ (absolute value of x)

The reason is:

(1) The function $f(x) = 1/x$ is defined for $x \neq 0$, so we would like to find a anti-derivative $F(x)$, i.e. a fn such that $F'(x) = 1/x$, that is also defined for all $x \neq 0$.

(2) $(\ln x)' = 1/x$ so $F(x) = \ln x$ is an anti-derivative, but it is only defined for $x > 0$

$$\int 1/x \, dx = \begin{cases} \ln x + C, & x > 0 \\ ? & x < 0 \end{cases}$$

(3) When $x < 0$, then $-x > 0$ and $\ln(-x)$ is defined, with

$$(\ln(-x))' = \frac{1}{-x} \cdot (-1) = \frac{1}{x} \quad \text{(by the chain rule)}$$

So

$$\int 1/x \, dx = \begin{cases} \ln x + C, & x > 0 \\ \ln(-x) + C, & x < 0 \end{cases}$$
$$= \ln|x| + C$$

iii) Integration of rational functions (fractions)

Ex: $\displaystyle\int \frac{x^2-1}{x+4}\,dx = \ ?$

(a) Use polynomial division if the degree of the numerator is at least as high as the degree of the denominator (i.e., whenever possible):

Ex: $\displaystyle\frac{x^2-1}{x+4} = x-4 + \frac{15}{x+4}$

$$
\begin{array}{l}
x^2 - 1 \ : \ x+4 = x-4 \quad \text{quotient} \\
-(x^2+4x) \\
\hline
\quad -4x - 1 \\
-(-4x-16) \\
\hline
\qquad 15 \quad \text{Remainder}
\end{array}
$$

$$\int \frac{x^2-1}{x+4}\,dx = \int x-4 + \frac{15}{x+4}\,dx = \frac{1}{2}x^2 - 4x + 15\int \frac{1}{x+4}\,dx$$

use subst. $\boxed{\begin{array}{l} u = x+4 \\ du = dx \end{array}}$

$$= \frac{1}{2}x^2 - 4x + 15\ln|x+4| + C$$

$\left(\int \frac{1}{u}\,du = \ln|u|\right)$

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b) When denominator has degree one, use substitution:

$$\int \frac{A}{ax+b}\,dx = \underline{\underline{\frac{A}{a}\ln|ax+b| + C}}$$

$\underline{\text{subst:}}$ $\boxed{\begin{array}{l} u = ax+b \\ du = a\cdot dx \end{array}}$ $\leftrightarrow$ $\left\{ \int \frac{A}{u}\frac{du}{a} = \frac{A}{a}\int \frac{1}{u}\,du = \frac{A}{a}\ln|u| + C \right.$

c) If the denominator has degree >1, use <u>partial fractions:</u>

$\underline{\text{Ex:}}$ $\int \frac{1}{x^2-4}\,dx$

$$= \int \frac{1/4}{x-2} + \frac{1/4}{x+2}\,dx$$

$$= \underline{\underline{\frac{1}{4}\ln|x-2| - \frac{1}{4}\ln|x+2| + C}}$$

<u>Partial fractions:</u>

since $x^2-4 = (x-2)(x+2)$ we try

$$\frac{1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2}$$

for const. A, B

$$\frac{1}{x^2-4} = \frac{A}{x-2} + \frac{B}{x+2} \quad \Big|\cdot(x^2-4)$$

$$1 = A\cdot(x+2) + B(x-2)$$

$$1 = \underbrace{(A+B)}_{=0}x + \underbrace{(2A-2B)}_{=1}$$

$B = -A$, $4A = 1$

$\underline{A = 1/4}$ $\underline{B = -1/4}$