BI HANDELSHØYSKOLEN
Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo
Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500
www.bi.no
info@bi.no
Org. nr. 971 22 8865

LECTURE 6

FORK 1003
MATHEMATICS

EIVIND ERIKSEN

AUG 7, 2017

Plan:

1. Partial derivatives and the Hessian matrix
2. Unconstrained max/min-problems
3. Lagrange problems

Review: Integration

$\int f(x)\,dx = F(x) + C$ $F(x)$: antiderivative ($F'(x) = f(x)$)

Computation:

- integration rules

$$\boxed{\begin{aligned} &\int (x^n)\,dx = \frac{1}{n+1}x^{n+1} + C, \quad n \neq -1 \\ &\int \frac{1}{x}\,dx = \ln|x| + C \\ &\int e^x\,dx = e^x + C \end{aligned}}$$

- Integration techniques:

i) Integration by parts: $\int u'v\,dx = uv - \int uv'\,dx$

Ex: $I = \int \frac{\ln x}{x}\,dx = \int \frac{1}{x}\cdot \ln x\,dx$.

$\boxed{\begin{matrix} u = \ln x & v = \ln x \\ u' = 1/x & v' = 1/x \end{matrix}}$

$= \ln x \cdot \ln x - \int \ln x \cdot \frac{1}{x}\,dx$

$I = (\ln x)^2 - I$

$2I = (\ln x)^2 \Rightarrow I = \frac{1}{2}(\ln x)^2 + C$ //

BI HANDELSHØYSKOLEN
Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo
Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500
www.bi.no
info@bi.no
Org. nr. 971 22 8865

ii) Substitution:

$$\boxed{\begin{aligned} u &= u(x) \\ du &= u'(x)\,dx \end{aligned}}$$

Ex:

$$\int \frac{2x}{x^2+1}\,dx = \int \frac{\cancel{2x}}{u}\cdot\frac{du}{\cancel{2x}} = \int \frac{1}{u}\,du$$

$$\boxed{\begin{aligned} u &= x^2+1 \\ du &= 2x\cdot dx \end{aligned}}$$

$$= \ln|u| + C = \ln|x^2+1| + C = \ln(x^2+1) + C$$

iii) Integration of rational functions.
- polynomial division
- partial fractions

a) $\int \frac{x}{x+1}\,dx$

$$\begin{array}{l} x : x+1 = 1 \\ -(x+1) \\ \hline -1 \end{array}$$

quot.

remainder

$$\int \frac{x}{x+1}\,dx = \int 1 + \frac{-1}{x+1}\,dx$$

$$= x - \int \frac{1}{x+1}\,dx$$

$$= x - \ln|x+1| + C$$

$$\boxed{\int \frac{A}{ax+b}\,dx = \frac{A}{a}\ln|ax+b| + C}$$

b) $\int \frac{x^2}{x^2-1}\,dx = \int \frac{x^2-1+1}{x^2-1}\,dx$

$$= \int 1 + \frac{1}{x^2-1}\,dx$$

$$= x + \int \frac{1}{(x-1)(x+1)}\,dx$$

$$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} \quad \Big|\cdot(x-1)(x+1)$$

$$\boxed{1 = A(x+1) + B\cdot(x-1)} \quad \begin{array}{l} x=1 \\ x=-1 \end{array}$$

$$1 = \underset{0}{\underset{\|}{(A+B)}}\,x + \underset{1}{\underset{\|}{(A-B)}}$$

$B = -A$, $A - B = 2A = 1$

$B = -\frac{1}{2}$ $A = \frac{1}{2}$

$$= x + \int \frac{1/2}{x-1} - \frac{1/2}{x+1}\,dx$$

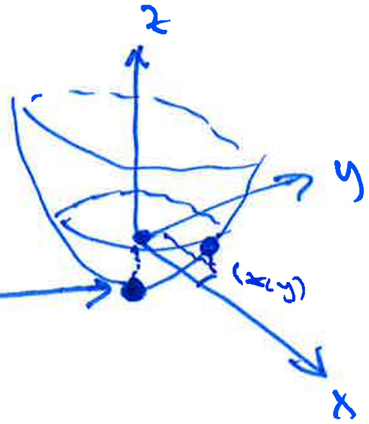
$$= x + \tfrac{1}{2}\ln|x-1| - \tfrac{1}{2}\ln|x+1| + C$$

$$= x + \frac{1}{2}\ln\frac{|x-1|}{|x+1|} + C$$

(1) Partial derivatives and the Hessian:

Ex: $f(x,y) = x^2 + 2y^2 - 1$

$x=0, y=0$: $f(0,0) = -1$
$(0,0,-1)$



$f(x,y)$: fn. in two var. $\qquad z = f(x,y)$

$\Rightarrow$ graph of f = all pts (x,y,z)
s.t. $z = f(x,y)$ for $(x,y) \in D_f$

= surface in 3-dim space

Partial derivatives: $\qquad$ **Ex:** $f(x,y) = x^2 + 2y^2 - 1$

$f'_x(x,y) = \underline{2x}$ ← keep y constant

$f'_y(x,y) = \underline{4y}$ ← keep x constant

Defn:

$$f'_x = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$f'_y = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo
Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500
www.bi.no
info@bi.no
Org. nr. 971 22 8865

Ex: $f(x,y) = x^3 - 3xy + y^3$

$$f'_x = 3x^2 - (3xy)'_x = 3x^2 - 3y \cdot (x)'_x = 3x^2 - 3y$$

$$f'_y = 0 - 3x \cdot 1 + 3y^2 = -3x + 3y^2$$

The Hessian matrix:

$$f''_{xx} = (3x^2 - 3y)'_x = 6x \qquad f''_{xy} = (3x^2 - 3y)'_y = -3$$

$$f''_{yx} = (-3x + 3y^2)'_x = -3 \qquad f''_{yy} = (-3x + 3y^2)'_y = 6y$$

Hessian matrix:

$$H(f)(x,y) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$$

Result: If f is "nice", then $H(f)(x,y)$ is symmetric.
("nice" = C^2 (f has cont. second order partial derivatives))

BI HANDELSHØYSKOLEN
Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo
Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500
www.bi.no
info@bi.no
Org. nr. 971 22 8865

2 Unconstrained optimization

$$\boxed{\max / \min \ f(x,y)}$$

Ex: $f(x,y) = x^3 - 3xy + y^3$

Stationary pt:
A stationary pt for f is a pt. (x,y) s.t. $f'_x = 0, f'_y = 0$

FOC = first order conditions

Result:
(x,y) is max/min
$\Downarrow$
(x,y) is stationary

$f'_x = 3x^2 - 3y = 0$
$f'_y = -3x + 3y^2 = 0$

$x^2 = y$
$-3x + 3(x^2)^2 = 0$
$-3x + 3x^4 = 0$
$-3x \cdot (1 - x^3) = 0$
$x = 0$ or $x^3 = 1$
$y = 0$ $\quad$ $x = 1$
$\quad\quad\quad\;\; y = 1$

Stat pts for f:
$(x,y) = (0,0), (1,1)$

Classification of stat pts as local max, local min, or saddle pts.

Second derivative test:

If (x^*, y^*) is a stationary pt with Hessian matrix

$$H(f)(x^*, y^*) = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

then:

$AC - B^2 > 0$, $A > 0 \Rightarrow (x^*, y^*)$ is local min

$AC - B^2 > 0$, $A < 0 \Rightarrow (x^*, y^*)$ is local max

$AC - B^2 < 0 \Rightarrow (x^*, y^*)$ is saddle pt

Comments:

1. $AC - B^2 = \det H(f)(x^*, y^*)$
2. $AC - B^2 = 0$: no conclusion (all other cases are covered)

Ex: $f(x,y) = x^3 - 3xy + y^3$

$$\left.\begin{aligned} f'_x &= 3x^2 - 3y = 0 \\ f'_y &= -3x + 3y^2 = 0 \end{aligned}\right\} \Rightarrow \text{stat. pts. } (0,0),\ (1,1)$$

Classification: $H(f)(x,y) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$

(0,0):

$H(f)(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$: $\det H = 0 - 9 = -9 < 0$

(0,0) saddle pt

(1,1):

$H(f)(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix}$: $\det H = 36 - 9 = 27 > 0$

$A = 6 > 0$

(1,1) local min

BI HANDELSHØYSKOLEN

Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo

Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500

www.bi.no
info@bi.no
Org. nr. 971 22 8865

Global max/min = max/min

Ex: max/min $f(x,y) = x^3 - 3xy + y^3$

Stationary pts: $\begin{cases} (0,0): & \text{saddle pt} \\ (1,1): & \text{local min} \end{cases}$

no max: (f → ∞)

no local max

min?

Cand: (1,1) with $f(1,1) = -1$

since & $f(-2,0) = -8$)

no min.

Method: Unconstrained optimization $\left\{ \begin{matrix} \max \\ \min \end{matrix} f(x,y) \right\}$

1. Find stationary pts using FOC: $\begin{cases} f'_x = 0 \\ f'_y = 0 \end{cases}$
2. Classify stationary pts using second derivative test
3. Determine if candidate pts (local max /min) (if possible) are global max/min.

If $|H(f)(x^*,y^*)| = 0$, and second derivative test is inconclusive, we must test for local max/min using the defn.

3 Constrained optimization and Lagrange problems

Lagrange problem: $\max/\min\ f(x,y)$ when $g(x,y)=a$

$=$ equality constraint

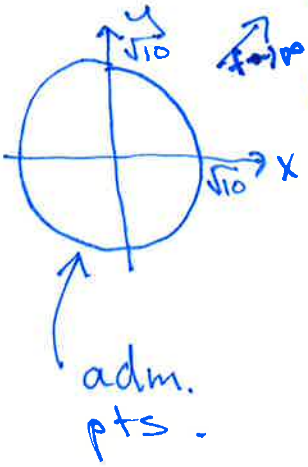
$f(x,y)$: objective function

$g(x,y)=a$: constraint

admissible pts

adm. pts = pts that satisfy the equation

Ex: $\max\ \underbrace{x+3y}_{=f(x,y)}$ when $\underbrace{x^2+y^2}_{=g(x,y)}=\underbrace{10}_{=a}$



$x^2+y^2=10$: circle with center $(0,0)$ and radius $\sqrt{10}$

$$\boxed{x^2+y^2=r^2}$$

Lagrange's method: λ = Lagrange multiplier

$$\mathcal{L}(x,y;\lambda) = f(x,y) - \lambda\cdot g(x,y)$$
$$= x+3y-\lambda\cdot(x^2+y^2)$$

$x^2+y^2=10$

$x^2+y^2-10=0$

FOC+C: Lagrange conditions

$$\boxed{\text{FOC: } \mathcal{L}'_x = 0,\quad \mathcal{L}'_y = 0} \rightarrow \begin{aligned} f'_x - \lambda\cdot g'_x &= 0 \\ f'_y - \lambda\cdot g'_y &= 0 \end{aligned} \rightarrow \boxed{\begin{aligned} 1-\lambda\cdot 2x &= 0 \\ 3-\lambda\cdot 2y &= 0 \\ x^2+y^2 &= 10 \end{aligned}}$$

$$\boxed{\text{C: } g(x,y)=a}$$

BI HANDELSHØYSKOLEN
Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo
Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500
www.bi.no
info@bi.no
Org. nr. 971 22 8865

Solutions of FOC+C are essentially the candidates for max/min in the Lagrange problem.

Ex: $1-\lambda \cdot 2x = 0$ $\qquad x = \frac{1}{2\lambda}$

$3-\lambda \cdot 2y = 0$ $\qquad y = 3/(2\lambda)$

$x^2+y^2 = 10$ $\qquad \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 = 10$

$\frac{1}{4\lambda^2} + \frac{9}{4\lambda^2} = 10 \qquad | \cdot 4\lambda^2$

$\frac{10}{40} = \frac{\cancel{10} \cdot 4\lambda^2}{40}$

$f(x,y) = x+3y$

$\lambda^2 = 1/4 \qquad \lambda = \pm 1/2$

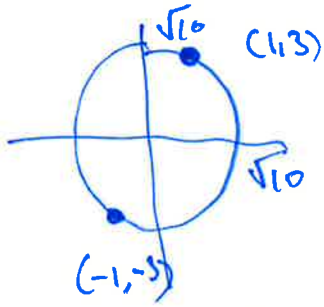
$\lambda = 1/2$:

$x=1, y=3, \lambda = 1/2$

$\lambda = -1/2$:

$x=-1, y=-3, \lambda = -1/2$

Solution of FOC+C = Candidates for max/min



$f(1,3) = 10$
max?

$f(-1,-3) = -10$
min?

Lagrange's thm: Necessary cond. for Lagrange problems

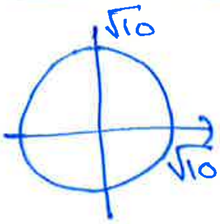
If (x^*, y^*) is a max/min in $\boxed{\text{max/min } f(x,y) \text{ when } g(x,y) = a}$

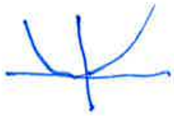
then either

i) there exists a λ s.t. $(x^*, y^*; \lambda)$ satisfies FOC+C, or

ii) $g'_x(x^*, y^*) = g'_y(x^*, y^*) = 0$ (degenerate constraint)

Ex: max $x + 3y$ when $x^2 + y^2 = 10$





DC: $g = x^2 + y^2$

$g'_x = 2x = 0$ $\quad$ $g'_y = 2y = 0$

$\underline{x = 0}$ $\quad$ $\underline{y = 0}$

$(0,0)$ is not adm. since $0^2 + 0^2 \neq 10$

Concl: no adm pt have DC

That means: (x^*, y^*) max $\Rightarrow$ $\underset{f=10}{(x^*, y^*) = (1,3)}$ or $\cancel{(-1,-3)}$ $\cancel{f=-10}$

EVT: $(x,y) = (1,3)$ is the maximum

Extreme value theorem:

If f is continuous fn that is defined on a closed and bounded set, then it has a max and a min.

BI HANDELSHØYSKOLEN

Nydalsveien 37, 0484 Oslo
Postadresse 0442 Oslo

Telefon 06600
Telefaks 21 04 80 00
Studieinfo. tlf. 810 00 500

www.bi.no
info@bi.no
Org. nr. 971 22 8865

In EVT: * All "nice" functions (the ones we will use in the course) are continuous

* An equality constraint $g(x,y)=a$ is closed

* Important condition:

Is the set of adm. pts $\{(x,y): g(x,y)=a\}$ bounded?

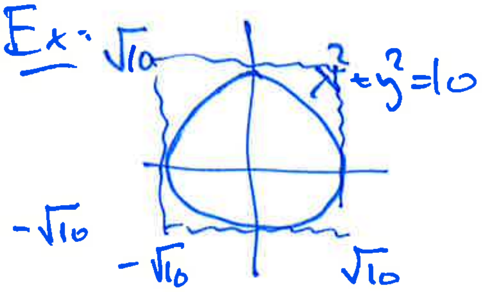
bounded $\Leftrightarrow$ there exist finite bounds m, M, n, N s.t.

$m \leq x \leq M$

$n \leq y \leq N$

for all (x,y) that are adm.

Ex:



The circle is bounded since

$-\sqrt{10} \leq x \leq \sqrt{10}$

$-\sqrt{10} \leq y \leq \sqrt{10}$

for all (x,y) on the circle $x^2+y^2=10$.