Plan

Economics/econometrics:

– video lectures

① Introduction:

Mathematics – 6 lectures

Morning: Matrices

Afternoon: Derivation, integration

② Solving linear systems

Ex:

$$\begin{aligned} 2x - 3y &= 2 \\ x + 4y &= 16 \end{aligned}$$

2×2 lin. system

equ's variables

$$\begin{aligned} x - y - 2z + w &= 3 \\ 2x + y + z - w &= 6 \\ 4x - 2y + 3z + 4w &= 12 \end{aligned}$$

3×4 lin. system

Solution techniques:

substitution methods

elimination methods

Substitution

Ex: $2x-3y=2$

$x+4y=16 \longrightarrow x=16-4y$

$2(16-4y)-3y=2$

$32-8y-3y=2$

$\frac{-11y}{-11}=\frac{-30}{-11}$ $\quad y=30/11$

$x=16-4\cdot\frac{30}{11}$

$=\frac{16\cdot 11-4\cdot 30}{11}$

$=56/11$

Elimination

Ex: $2x-3y=2$

$x+4y=16 \quad |\cdot 2$

$\downarrow$

I $\quad 2x-3y=2 \quad -1$

II $\quad 2x+8y=32 \quad +$

$\downarrow$

I $\quad 2x-3y=2$

II-I $\quad 11y=30$

$2x-3\cdot(30/11)=2$

$y=30/11$

$2x=\frac{90}{11}+2=\frac{90+2\cdot 11}{11}$

$x=\frac{112}{11}:2=\cancel{\frac{112}{22}}\ \frac{56}{11}$

What characterizes linear systems?

Linear equation: $a_1 x_1 + a_2 x_2 + \dots + a_n x_n = b$

($a_1, a_2, \dots, a_n, b$: given numbers
$x_1, x_2, \dots, x_n$: variables)

Ex:

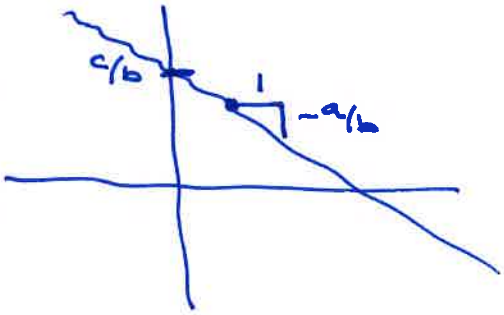
$2x - 3y = z$ linear ✓

$2x - 3y - z = 0$

$\sqrt{x} + 3y = 7$ not linear

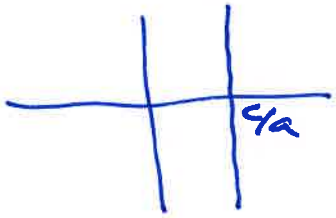
$xy = 1$ — " —

Geometry: $n = 2$ $ax + by = c$
a straight line



$b \neq 0$: $\dfrac{by}{b} = \dfrac{c - ax}{b}$

$$y = \frac{c}{b} - \frac{a}{b} \cdot x$$

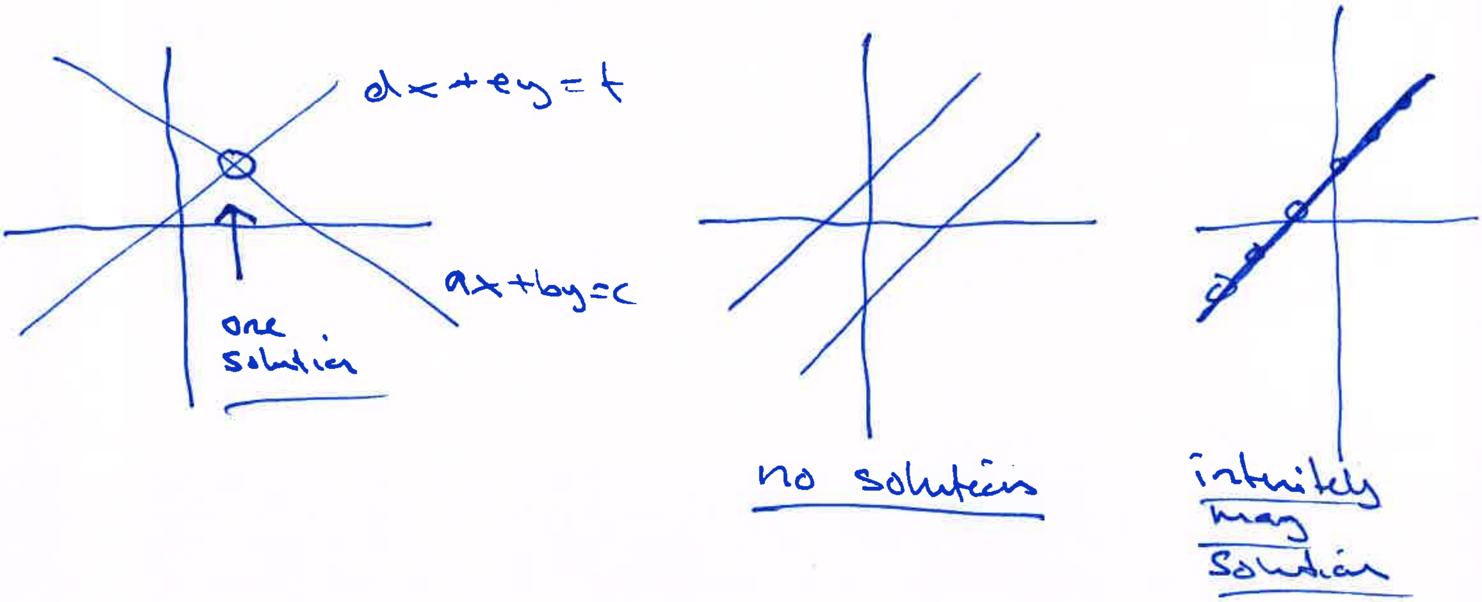


$b = 0$: $ax = c$
$a \neq 0$: $x = c/a$

[$a = b = 0$: $0 \cdot x + 0 \cdot y = c$
(degenerate case)]

2x2 lin. system:

$$ax + by = c$$
$$dx + ey = f$$



Note:

i) Linear equations give graphs that do not curve

ii) Linear systems: solutions are either

i) exactly one solution

ii) infinitely many solutions

iii) no solution

cannot have exactly two solutions

3 Gaussian elimination.

- general method for solving any linear system
- fast – instructive

Ex:

$$
\begin{aligned}
x+y+z &= 3 \\
x+2y+4z &= 7 \\
x+3y+9z &= 13
\end{aligned}
$$

3×3 lin. system

Coefficient matrix:

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$$

Augmented matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right)$$

Elementary row operations:

i) Switch two rows
ii) Multiply a row with a number $c \neq 0$
iii) Add a multiple of one row to another row.

"allowed" operations, i.e the solutions are preserved

Ex:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right) \cdot(-1),\ + \quad \Leftrightarrow \quad \begin{cases} x+y+z=3 \\ x+2y+4z=7 \\ x+3y+9z=13 \end{cases}$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 1 & 3 & 9 & 13 \end{array}\right) -1,\ + \qquad II - I$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right) -2,\ + \qquad III - I$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right) \rightarrow \quad \begin{aligned} x+y+z &= 3 \\ y+3z &= 4 \\ 2z &= 2 \end{aligned}$$

echelon form
("staircase" / "triangular")

$$\Downarrow$$

$\underline{z=1}$

$y+3\cdot(1)=4 \quad \underline{y=1}$

$x+1+1=3 \quad \underline{x=1}$

One solution:

$x=1$
$y=1 \quad \Leftrightarrow \quad (x,y,z)=(1,1,1)$
$z=1$

Defn: the first (left-most) non-zero element in a row is called a leading coefficient or pivot.

Ex:

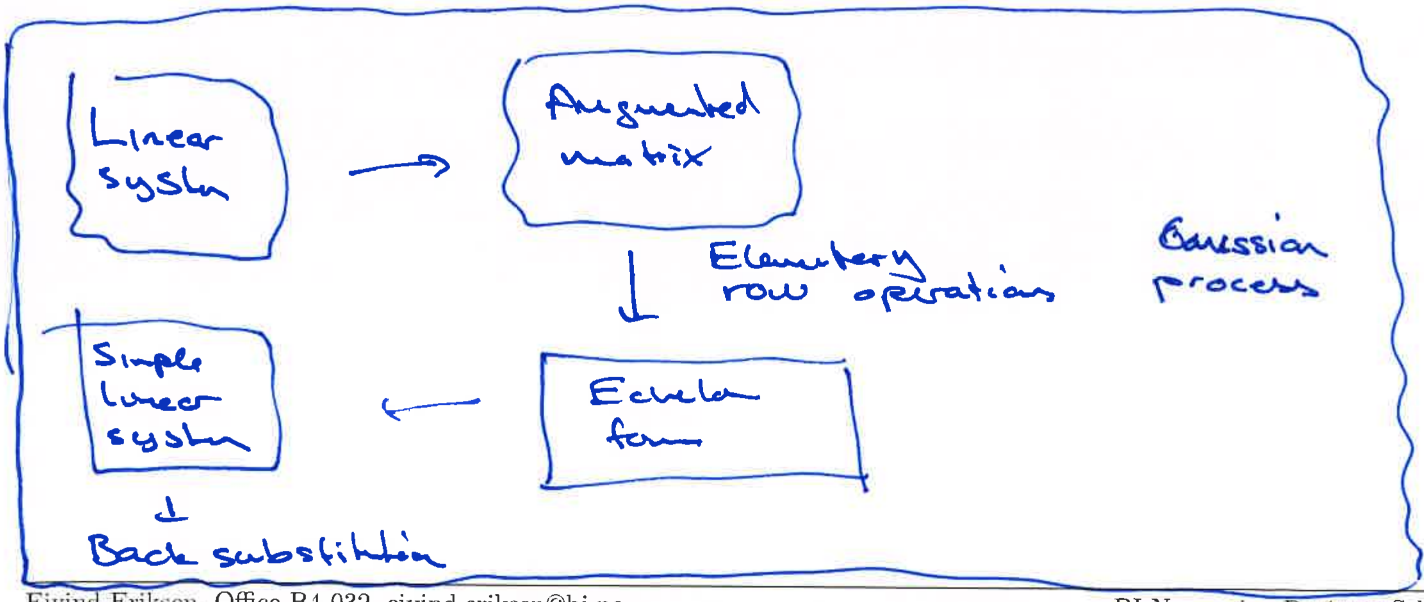
$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

circled: pivot or leading coeff.

Defn: A matrix is in echelon form if the following conditions are satisfied.

1) All zero rows should be in the bottom of the matrix (below non-zero rows)
2) All entries under a pivot are zero.

Back substitution: – solve in reverse order, and substitute the variables you have solved for.



More examples:

$$
\begin{aligned}
x + y + 4z - w &= 7\\
2x - y + z + 4w &= 3\\
x + 4y + 11z - 7w &= 15
\end{aligned}
$$

$$
\left(\begin{array}{cccc|c}
1 & 1 & 4 & -1 & 7\\
2 & -1 & 1 & 4 & 3\\
1 & 4 & 11 & -7 & 15
\end{array}\right)
\begin{array}{l} -2 \quad -1\\ + \\ \quad\;\; + \end{array}
$$

$$\downarrow$$

$$
\left(\begin{array}{cccc|c}
1 & 1 & 4 & -1 & 7\\
0 & -3 & -7 & 6 & -11\\
0 & 3 & 7 & -6 & 8
\end{array}\right)
\begin{array}{l} \\ 1 \\ + \end{array}
$$

$$\downarrow$$

$$
\left(\begin{array}{cccc|c}
1 & 1 & 4 & -1 & 7\\
0 & -3 & -7 & 6 & -11\\
0 & 0 & 0 & 0 & -3
\end{array}\right)
$$

echelon form

$$
\begin{aligned}
x + y + 4z - w &= 7\\
-3y - 7z + 6w &= -11\\
\boxed{0\cdot x + 0\cdot y + 0\cdot z + 0\cdot w = -3}
\end{aligned}
$$

$$\Updownarrow$$

$$0 = -3$$

no solutions

In general:

no solutions $\iff$ pivot in the last column

Ex: $x + y + 4z - w = 7$
$2x - y + z + 4w = 3$
$x + 4y + 11z - 7w = 18$

$$\left(\begin{array}{cccc|c} 1 & 1 & 4 & -1 & 7 \\ 2 & -1 & 1 & 4 & 3 \\ 1 & 4 & 11 & -7 & 18 \end{array}\right) \begin{array}{l} -2 \quad -1 \\ + \\ \quad\quad + \end{array}$$

$\downarrow$

$$\left(\begin{array}{cccc|c} 1 & 1 & 4 & -1 & 7 \\ 0 & -3 & -7 & 6 & -11 \\ 0 & 3 & 7 & -6 & 11 \end{array}\right) \begin{array}{l} \\ 1 \\ + \end{array}$$

$\downarrow$

$$\left(\begin{array}{cccc|c} 1 & 1 & 4 & -1 & 7 \\ 0 & -3 & -7 & 6 & -11 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

echelon form

$$\boxed{\begin{array}{l} x + y + 4z - w = 7 \\ -3y - 7z + 6w = -11 \end{array}}$$

$$\boxed{\cancel{0 \cdot x + 0 \cdot y + 0 \cdot z + 0 \cdot w = 0}}$$

$0 = 0$

II $-3y - 7z + 6w = -11$

$$\frac{-3y}{-3} = \frac{7z - 6w - 11}{-3} \qquad y = -\frac{7}{3}z + 2w + \frac{11}{3}$$

I $x + \left(-\frac{7}{3}z + 2w + \frac{11}{3}\right) + 4z - w = 7$

$$x = \frac{7}{3}z - 2w - \frac{11}{3} - 4z + w + 7$$

$$x = -\frac{5}{3}z - w + \frac{10}{3}$$

$$x = -\frac{5}{3}z - w + \frac{10}{3}$$
$$y = -\frac{7}{3}z + 2w + \frac{11}{3}$$

} basic variables

$$z = z$$
$$w = w$$

} free variables

$$(x, y, z, w) = \left(-\frac{5}{3}z - w + \frac{10}{3}, -\frac{7}{3}z + 2w + \frac{11}{3}, z, w\right)$$

where z, w are free

infinitely many solutions,
two degrees of freedom (two free var's)

In general:
If the linear system has solutions (no pivot in the last column)

then:

$$\boxed{\begin{array}{l} \text{free variables — columns without pivots} \\ \text{basic variables — columns with pivots} \end{array}}$$

No free variables: one solution

At least one free variable: infinitely many solutions

Ex:
$$3x + 4y - z = 7$$
$$7x - 5y + z = 0$$

$$\left(\begin{array}{ccc|c} 3 & 4 & -1 & 7 \\ 7 & -5 & 1 & 0 \end{array}\right) \begin{array}{l} -7/3 \\ + \end{array}$$

$$\vdots$$

Alt:
$$\left(\begin{array}{ccc|c} 3 & 4 & -1 & 7 \\ 7 & -5 & 1 & 0 \end{array}\right) \begin{array}{l} \cdot 7 \\ \cdot 3 \end{array}$$
$$\downarrow$$
$$\left(\begin{array}{ccc|c} 21 & 28 & -7 & 49 \\ 21 & -15 & 3 & 0 \end{array}\right) \begin{array}{l} -1 \\ + \end{array}$$
$$\downarrow$$
$$\left(\begin{array}{ccc|c} 21 & 28 & -7 & 49 \\ 0 & -43 & 10 & -49 \end{array}\right)$$
echelon form

Alt:
$$3x + 4y - z = 7$$
$$7x - 5y + z = 0$$

$$-z + 4y + 3x = 7$$
$$z - 5y + 7x = 0$$

$$\left(\begin{array}{ccc|c} -1 & 4 & 3 & 7 \\ 1 & -5 & 7 & 0 \end{array}\right) \begin{array}{l} 1 \\ + \end{array}$$

$$\left(\begin{array}{ccc|c} -1 & 4 & 3 & 7 \\ 0 & -1 & 10 & 7 \end{array}\right)$$
echelon form

$$-\cancel{z} + 4y + 3x = 7$$
$$-y + 10x = 7$$

$$y = 10x - 7$$
$$z = 4(10x-7) + 3x - 7$$
$$= 43x - 35$$

Ex:

$\cancel{2x + 3y - z = 7}$

$\cancel{4x \quad + 3z =}$

$$\boxed{\begin{aligned} 3y - z &= 7 \\ 4x + y + z &= 1 \\ x - y + 4z &= 12 \end{aligned}}$$

$$\left(\begin{array}{ccc|c} 0 & 3 & -1 & 7 \\ 4 & 1 & 1 & 1 \\ 1 & -1 & 4 & 12 \end{array}\right)$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & -1 & 4 & 12 \\ 4 & 1 & 1 & 1 \\ 0 & 3 & -1 & 7 \end{array}\right)$$

we can switch two rows to get a pivot in upper left corner nonzero coefficient is 1:

$$
\begin{aligned}
x_1 - 0.4x_2 - 0.3\ x_3 &= 130 \\
x_2 - 0.25x_3 &= 125 \\
x_3 &= 300.
\end{aligned}
\tag{11}
$$

Now, instead of using back substitution, use Gaussian elimination methods from the *bottom* equation to the top to eliminate all but the first term on the left-hand side in each equation in (11). For example, add 0.25 times equation (11c) to equation (11b) to eliminate the coefficient of x_3 in (11b) and obtain $x_2 = 200$. Then, add 0.3 times (11c) to (11a) and 0.4 times (11b) to (11a) to obtain the new system:

$$
\begin{aligned}
x_1 \qquad\qquad &= 300 \\
x_2 \qquad &= 200 \\
x_3 &= 300,
\end{aligned}
\tag{12}
$$

which needs no further work to see the solution. Gauss-Jordan elimination is particularly useful in developing the theory of linear systems; Gaussian elimination is usually more efficient in solving actual linear systems.

Earlier we mentioned a third method for solving linear systems, namely matrix methods. We will study these methods in the next two chapters, when we discuss matrix inversion and Cramer's rule. For now, it suffices to note that all the intuition behind these more advanced methods derives from Gaussian elimination. The understanding of this technique will provide a solid base on which to build your knowledge of linear algebra.

EXERCISES

7.1 Which of the following equations are linear?

a) $3x_1 - 4x_2 + 5x_3 = 6$; *b)* $x_1x_2x_3 = -2$; *c)* $x^2 + 6y = 1$;
d) $(x + y)(x - z) = -7$; *e)* $x + 3^{1/2}z = 4$; *f)* $x + 3z^{1/2} = -4$.

7.2 Solve the following systems by substitution, Gaussian elimination, and Gauss-Jordan elimination:

a)
$$
\begin{aligned}
x - 3y + 6z &= -1 \\
2x - 5y + 10z &= 0 \\
3x - 8y + 17z &= 1;
\end{aligned}
$$

b)
$$
\begin{aligned}
x_1 + x_2 + x_3 &= 0 \\
12x_1 + 2x_2 - 3x_3 &= 5 \\
3x_1 + 4x_2 + x_3 &= -4.
\end{aligned}
$$

7.3 Solve the following systems by Gauss-Jordan elimination. Note that the third system requires an equation interchange.

a) $3x + 3y = 4$
$x - y = 10;$

b) $4x + 2y - 3z = 1$
$6x + 3y - 5z = 0$
$x + y + 2z = 9;$

c) $2x + 2y - z = 2$
$x + y + z = -2$
$2x - 4y + 3z = 0.$

7.4 Formalize the three elementary equation operations using the abstract notation of system (2), and for each operation, write out the operation which reverses its effect.

7.5 Solve the IS-LM system in Exercise 6.7 by substitution.

7.6 Consider the general IS-LM model with no fiscal policy in Chapter 6. Suppose that $M_s = M^o$; that is, the intercept of the LM-curve is 0.
a) Use substitution to solve this system for Y and r in terms of the other parameters.
b) How does the equilibrium GNP depend on the marginal propensity to save?
c) How does the equilibrium interest rate depend on the marginal propensity to save?

7.7 Use Gaussian elimination to solve

$$\begin{cases} 3x + 3y = 4 \\ -x - y = 10. \end{cases}$$

What happens and why?

7.8 Solve the general system

$$\begin{cases} a_{11}x_1 + a_{12}x_2 = b_1 \\ a_{21}x_1 + a_{22}x_2 = b_2. \end{cases}$$

What assumptions do you have to make about the coefficients a_{ij} in order to find a solution?

7.2 ELEMENTARY ROW OPERATIONS

The focus of our concern in the last section was on the coefficients a_{ij} and b_i of the systems with which we worked. In fact, it was a little inefficient to rewrite the x_i's, the plus signs, and the equal signs each time we transformed a system. It makes sense to simplify the representation of linear system (2) by writing two rectangular arrays of its coefficients, called **matrices**. The first array is

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

7.11 Write the three systems in Exercise 7.3 in matrix form. Then use row operations to find their corresponding row echelon and reduced row echelon forms and to find the solution.

7.12 Use Gauss-Jordan elimination in matrix form to solve the system

$$\begin{aligned} w + x + 3y - 2z &= 0 \\ 2w + 3x + 7y - 2z &= 9 \\ 3w + 5x + 13y - 9z &= 1 \\ -2w + x \qquad\quad - z &= 0. \end{aligned}$$

7.3 SYSTEMS WITH MANY OR NO SOLUTIONS

As we will study in more detail later, the locus of all points (x_1, x_2) which satisfy the linear equation $a_{11}x_1 + a_{12}x_2 = b_1$ is a straight line in the plane. Therefore, the solution (x_1, x_2) of the two linear equations in two unknowns

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= b_1 \\ a_{21}x_1 + a_{22}x_2 &= b_2 \end{aligned} \tag{16}$$

is a point which lies on both lines of (16) in the Cartesian plane. Solving system (16) is equivalent to finding where the two lines given by (16) cross. In general, two lines in the plane will be nonparallel and will cross in exactly one point. However, the lines given by (16) can be parallel to each other. In this case, they will either coincide or they will never cross. If they coincide, every point on either line is a solution to (16); and (16) has *infinitely* many solutions. An example is the system

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ 2x_1 + 4x_2 &= 6. \end{aligned}$$

In the case where the two parallel lines do not cross, the corresponding system has *no* solution, as the example

$$\begin{aligned} x_1 + 2x_2 &= 3 \\ x_1 + 2x_2 &= 4 \end{aligned}$$

illustrates. Therefore, it follows from geometric considerations that two linear equations in two unknowns can have one solution, no solution, or infinitely many solutions. We will see later in this chapter that this principle holds for every system of *m linear* equations in *n* unknowns.

$$\left(\begin{array}{ccccccc|c} * & w & w & w & w & w & w & w \\ 0 & 0 & 0 & * & w & w & w & w \\ 0 & 0 & 0 & 0 & * & w & w & w \\ 0 & 0 & 0 & 0 & 0 & 0 & * & w \end{array}\right).$$

This matrix is in row echelon form. The corresponding reduced row echelon form is

$$\left(\begin{array}{ccccccc|c} 1 & w & w & 0 & 0 & w & 0 & w \\ 0 & 0 & 0 & 1 & 0 & w & 0 & w \\ 0 & 0 & 0 & 0 & 1 & w & 0 & w \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & w \end{array}\right).$$

The final solution will have the form

$$\begin{aligned} x_1 &= a_1 - a_2x_2 - a_3x_3 - a_4x_6, \\ x_4 &= b_1 \qquad\qquad\qquad - b_2x_6, \\ x_5 &= c_1 \qquad\qquad\qquad - c_2x_6, \\ x_7 &= d_1. \end{aligned}$$

Here x_7 is the only variable which is unambiguously determined. The variables x_2, x_3, and x_6 are free to take on any values; once values have been selected for these three variables, then values for x_1, x_4, and x_5 are automatically determined.

Some more vocabulary is helpful here. If the jth column of the row echelon matrix $\hat{B}$ contains a pivot, we call x_j a **basic variable.** If the jth column of $\hat{B}$ does not contain a pivot, we call x_j a **free** or **nonbasic variable.** In this terminology, Gauss-Jordan elimination determines a solution of the system in which each basic variable is either unambiguously determined or a linear expression of the free variables. The free variables are free to take on any value. Once one chooses values for the free variables, values for the basic variables are determined.

As in the example above, the free variables are often placed on the right-hand side of the equations to emphasize that their values are not determined by the system; rather, they act as parameters in determining values for the basic variables.

In a given problem which variables are free and which are basic may depend on the order of the operations used in the Gaussian elimination process and on the order in which the variables are indexed.

EXERCISES

7.13 Reduce the following matrices to row echelon and reduced row echelon forms:

a) $\begin{pmatrix} 1 & 1 \\ -2 & -1 \end{pmatrix}$, *b)* $\begin{pmatrix} 1 & 3 & 4 \\ 2 & 5 & 7 \end{pmatrix}$, *c)* $\begin{pmatrix} -1 & -1 \\ 2 & 1 \\ 1 & 0 \end{pmatrix}$.

7.14 Solve the system of equations $\begin{cases} -4x + 6y + 4z = 4 \\ 2x - y + z = 1. \end{cases}$

7.15 Use Gauss-Jordan elimination to determine for what values of the parameter k the system

$$\begin{aligned} x_1 + x_2 &= 1 \\ x_1 - kx_2 &= 1 \end{aligned}$$

has no solutions, one solution, and more than one solution.

7.16 Use Gauss-Jordan elimination to solve the following four systems of linear equations. Which variables are free and which are basic in each solution?

$$a)\quad \begin{aligned} w + 2x + y - z &= 1 \\ 3w - x - y + 2z &= 3 \\ -x + y - z &= 1 \\ 2w + 3x + 3y - 3z &= 3; \end{aligned} \qquad b)\quad \begin{aligned} w - x + 3y - z &= 0 \\ w + 4x - y + z &= 3 \\ 3w + 7x + y + z &= 6 \\ 3w + 2x + 5y - z &= 3; \end{aligned}$$

$$c)\quad \begin{aligned} w + 2x + 3y - z &= 1 \\ -w + x + 2y + 3z &= 2 \\ 3w - x + y + 2z &= 2 \\ 2w + 3x - y + z &= 1; \end{aligned} \qquad d)\quad \begin{aligned} w + x - y + 2z &= 3 \\ 2w + 2x - 2y + 4z &= 6 \\ -3w - 3x + 3y - 6z &= -9 \\ -2w - 2x + 2y - 4z &= -6. \end{aligned}$$

7.17 *a)* Use the flexibility of the free variable to find *positive integers* which satisfy the system

$$\begin{aligned} x + y + z &= 13 \\ x + 5y + 10z &= 61. \end{aligned}$$

b) Suppose you hand a cashier a dollar bill for a 6-cent piece of candy and receive 16 coins as your change — all pennies, nickels, and dimes. How many coins of each type do you receive? [Hint: See part *a*.]

7.18 For what values of the parameter a does the following system of equations have a solution?

$$\begin{aligned} 6x + y &= 7 \\ 3x + y &= 4 \\ -6x - 2y &= a. \end{aligned}$$

7.19 From Chapter 6, the stationary distribution in the Markov model of unemployment satisfies the linear system

$$\begin{aligned} (q-1)x + py &= 0 \\ (1-q)x - py &= 0 \\ x + y &= 1. \end{aligned}$$