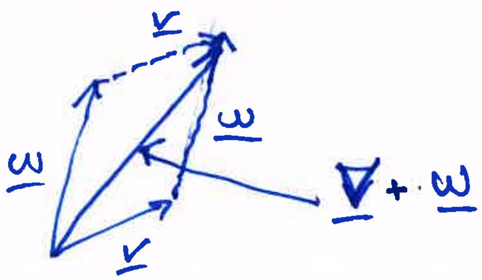
Plan

(1) Vectors and matrices

vector: quantity that has a direction and a magnitude

$\|\underline{v}\|$: length of the vector $\underline{v}$, represents the magnitude

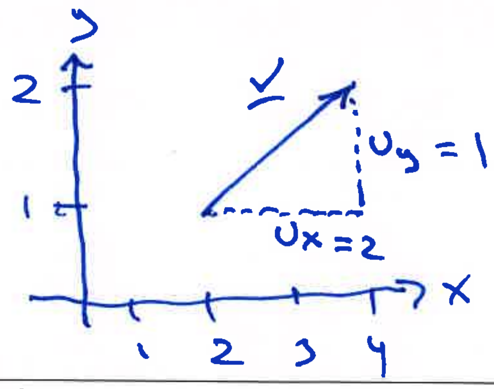


Coordinates:

$\underline{v} = (v_x, v_y)$

$= (2, 1)$

$= \begin{pmatrix} 2 \\ 1 \end{pmatrix}$



$\|\underline{v}\| = \sqrt{v_x^2 + v_y^2}$

$= \sqrt{2^2 + 1^2}$

$= \sqrt{5}$

n-vector: $\underline{v} = (v_1, v_2, \ldots, v_n) = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

column vector

Operations on vectors: $\underline{v} = (v_1, v_2, \ldots, v_n)$, $\underline{w} = (w_1, w_2, \ldots, w_n)$

addition: $\underline{v} + \underline{w} = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$

subtraction: $\underline{v} - \underline{w} = (v_1 - w_1, v_2 - w_2, \ldots, v_n - w_n)$

scalar multiplication: $r \cdot \underline{v} = (rv_1, rv_2, \ldots, rv_n)$

(scalar = number)

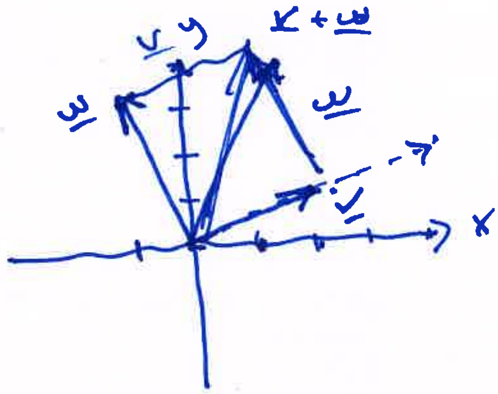
length: $\|\underline{v}\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$

Ex: $\underline{v} = (2,1)$ $\underline{w} = (-1,3)$

$\underline{v} + \underline{w} = (1,4)$

$\underline{v} - \underline{w} = (3,-2)$

$2 \cdot \underline{v} = (4,2)$



Matrices:

Defn: An $m \times n$ matrix A is a rectangular array of numbers, with m rows and n columns.

Ex: $A = \begin{pmatrix} 2 & 1 & 3 \\ 7 & -1 & 0 \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$

2×3-matrix

$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}$ m×n matr.

$B = \begin{pmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & \ddots & \vdots \\ b_{m1} & \cdots & b_{mn} \end{pmatrix}$ "

Operations on matrices:

- **addition:** $A + B$

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ \vdots & & \ddots & \vdots \\ a_{m1}+b_{m1} & \cdots & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

- **subtraction:** $A - B =$

$$\begin{pmatrix} a_{11}-b_{11} & \cdots & a_{1n}-b_{1n} \\ \vdots & \ddots & \\ a_{m1}-b_{m1} & \cdots & a_{mn}-b_{mn} \end{pmatrix}$$

- **Scalar multiplication:** $r \cdot A = \begin{pmatrix} ra_{11} & ra_{12} & \cdots & ra_{1n} \\ \vdots & & & \\ ra_{m1} & ra_{m2} & \cdots & ra_{mn} \end{pmatrix}$

Ex: $A = \begin{pmatrix} 2 & 1 & 3 \\ 0 & 1 & 7 \end{pmatrix}$ $B = \begin{pmatrix} 1 & 2 & -1 \\ 3 & 4 & 0 \end{pmatrix}$

$A + B = \begin{pmatrix} 3 & 3 & 2 \\ 3 & 5 & 7 \end{pmatrix}$ $A - B = \begin{pmatrix} 1 & -1 & 4 \\ -3 & -3 & 7 \end{pmatrix}$

$3A = \begin{pmatrix} 6 & 3 & 9 \\ 0 & 3 & 21 \end{pmatrix}$ $-B = \begin{pmatrix} -1 & -2 & 1 \\ -3 & -4 & 0 \end{pmatrix}$

Inner product: (dot product, scalar product)

Ex: $\underline{v} = (2,1)$ $\quad \underline{v}\cdot\underline{w} = 2\cdot(-1) + 1\cdot 3$

$\underline{w} = (-1,3)$ $\quad = 1$

$\langle \underline{v},\underline{w}\rangle = 1$

In general:

$\underline{v} = (v_1, v_2, \ldots, v_n)$ $\quad \underline{v}\cdot\underline{w} = v_1w_1 + v_2w_2 + \ldots + v_nw_n$

$\underline{w} = (w_1, w_2, \ldots, w_n)$ $\quad \langle \underline{v},\underline{w}\rangle = v_1w_1 + v_2w_2 + \ldots + v_nw_n$

Defn: $\underline{v} \perp \underline{w}$ = $\underline{v}$ is orthogonal to $\underline{w}$ means that the angle between $\underline{v}$ and $\underline{w}$ is a right angle (90°).

Result: $\boxed{\underline{v} \perp \underline{w} \text{ if and only if } \underline{v}\cdot\underline{w} = 0}$

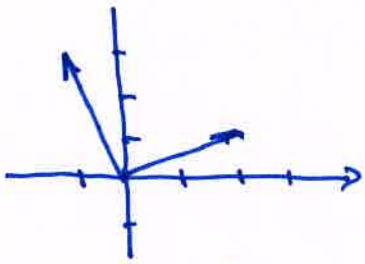
In fact: $\underline{v}\cdot\underline{w} > 0$: angle is less than 90°

$\underline{v}\cdot\underline{w} < 0$: angle is more than 90°

Ex: $\underline{v} = (2,1)$

$\underline{w} = (-1,3)$

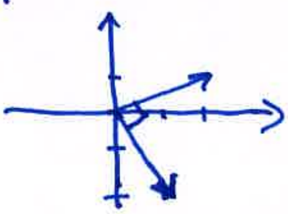
$\underline{v}\cdot\underline{w} = 1 > 0$



$\underline{v} = (2,1)$

$\underline{w} = (1,-2)$

$\underline{v}\cdot\underline{w} = 2-2 = 0$



(2) Matrix multiplication

A, B matrices $\qquad$ $A \cdot B$ matrix multiplication

Ex: $A = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix}$ $\quad B = \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix}$

$$A \cdot B = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix} \cdot \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ -4 & 0 \end{pmatrix}$$

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$$

$$A \cdot B = \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix}$$

$$\begin{array}{c|c} & \begin{pmatrix} 3 & 2 \\ -1 & 0 \end{pmatrix} = B \\ \hline A = \begin{pmatrix} 1 & 2 \\ 0 & 4 \end{pmatrix} & \begin{pmatrix} 1 & 2 \\ -4 & 0 \end{pmatrix} = AB \end{array}$$

In general:

A ($m \times n$), B ($n \times p$) $\rightsquigarrow$ $A \cdot B$ ($m \times p$)

$(AB)_{ij}$ = inner (dot) product of row i in A · col j in B

requirement: # rows in B = # cols in A

Notice: $A \cdot B \neq B \cdot A$

Ex:

$$A = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 3 & 2 & 4 \end{pmatrix} \qquad \underline{u} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$$

3×3 $\qquad$ 3×1

$$A \cdot \underline{u} = \begin{pmatrix} 2 & 1 & 3 \\ 1 & -1 & 2 \\ 3 & 2 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 9 \\ 4 \\ 13 \end{pmatrix}$$

Matrix multiplication and linear systems

$$\begin{aligned} x + 2y - z &= 7 \\ 4x - y + 2z &= 0 \\ 2x + y + z &= 4 \end{aligned} \quad \leadsto \quad \left(\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 4 & -1 & 2 & 0 \\ 2 & 1 & 1 & 4 \end{array}\right)$$

3x3 linear system

$$\boxed{A \cdot \underline{x} = \underline{b}}$$ matrix form of the linear system

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 4 & -1 & 2 \\ 2 & 1 & 1 \end{pmatrix} \qquad \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 7 \\ 0 \\ 4 \end{pmatrix}$$

coeff. matrix

$$A \cdot \underline{x} = \begin{pmatrix} 1 & 2 & -1 \\ 4 & -1 & 2 \\ 2 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x + 2y - z \\ 4x - y + 2z \\ 2x + y + z \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 4 \end{pmatrix}$$

Ex:
$$\begin{pmatrix}1&1\\0&0\end{pmatrix}\cdot\begin{pmatrix}0&1\\0&1\end{pmatrix}=\begin{pmatrix}0&2\\0&0\end{pmatrix}$$
$$\begin{pmatrix}0&1\\0&1\end{pmatrix}\cdot\begin{pmatrix}1&1\\0&0\end{pmatrix}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$$
$AB \neq BA$

Identity matrix:
$$I_2=\begin{pmatrix}1&0\\0&1\end{pmatrix}=I$$
$$I_3=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=I$$

Ex:
$$\begin{pmatrix}1&1\\0&0\end{pmatrix}\cdot\begin{pmatrix}1&0\\0&1\end{pmatrix}=\begin{pmatrix}1&1\\0&0\end{pmatrix}$$
$$A\cdot I = A$$

In general:
$$\begin{cases}A\cdot I=A\\ I\cdot A=A\end{cases}$$
for any matrix A

Powers:
A $n\times n$-matrix
$$A^2=A\cdot A$$
$$A^3=(A\cdot A)\cdot A$$
$$\vdots$$

Ex: $A=\begin{pmatrix}1&2\\1&4\end{pmatrix}$ 2×2
$$A^2=\begin{pmatrix}1&2\\1&4\end{pmatrix}\cdot\begin{pmatrix}1&2\\1&4\end{pmatrix}=\begin{pmatrix}3&10\\5&18\end{pmatrix}$$
2×2 2×2
$$A^3=A^2\cdot A=\begin{pmatrix}3&10\\5&18\end{pmatrix}\cdot\begin{pmatrix}1&2\\1&4\end{pmatrix}=\begin{pmatrix}13&46\\23&82\end{pmatrix}$$

③ Inverse matrices.

Linear system in matrix form: $A \cdot \underline{x} = \underline{b}$

~~$\underline{x} = \underline{b}/A$~~ $\underline{x} = A^{-1} \cdot b$

Defn: An inverse of a matrix A is a matrix A^{-1} such that $\left\{ \begin{matrix} A \cdot A^{-1} = I \\ A^{-1} \cdot A = I \end{matrix} \right\}$

Ex: $A = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}$

$A^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ s.t. $\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

Method: $(A|I) = \left(\begin{array}{cc|cc} ① & 2 & 1 & 0 \\ 1 & 4 & 0 & 1 \end{array}\right)]\cdot(-1) \rightarrow \left(\begin{array}{cc|cc} ① & 2 & 1 & 0 \\ 0 & ② & -1 & 1 \end{array}\right)]\cdot(-1)$

echelon form

$\rightarrow \left(\begin{array}{cc|cc} ① & 0 & 2 & -1 \\ 0 & ② & -1 & 1 \end{array}\right) \cdot \frac{1}{2}$

$\rightarrow \left(\begin{array}{cc|cc} ① & 0 & 2 & -1 \\ 0 & ① & -1/2 & 1/2 \end{array}\right) = (I \mid A^{-1})$

reduced echelon form

- echelon form s.t.
  - * all pivots are 1
  - * all entries over a pivot are zero

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} \qquad A^{-1} = \begin{pmatrix} 2 & -1 \\ -1/2 & 1/2 \end{pmatrix}$$

$$A \cdot A^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 2 & -1 \\ -1/2 & +1/2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \checkmark \text{ (ok)}$$

$$A^{-1} \cdot A = \begin{pmatrix} 2 & -1 \\ -1/2 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

In general:
- the inverse matrix does not always exist.
- if A^{-1} exists, then it is unique

(only non- matrices can have inverses)

Method: $(A|I) \to$ reduced echelon form

If the result is $(I|B)$, then $A^{-1} = B$

If the result is not $(I|B)$, then A^{-1} does not exist.

Ex: $A = \begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix}$

$$(A|I) = \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 2 & 4 & 0 & 1 \end{array}\right) \; \text{(}-2\text{)}$$

$$\to \left(\begin{array}{cc|cc} 1 & 2 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right)$$

A^{-1} does not exist.

Ex:

$$x + 2y = 7$$
$$x + 4y = 13$$

$$\begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7 \\ 13 \end{pmatrix} \quad \text{matrix form.}$$

$$A = \begin{pmatrix} 1 & 2 \\ 1 & 4 \end{pmatrix}$$

$$A^{-1} = \begin{pmatrix} 2 & -1 \\ -1/2 & 1/2 \end{pmatrix}$$

$$A \cdot \underline{x} = \underline{b} \quad | \; A^{-1} \cdot$$
$$A^{-1} \cdot A\underline{x} = A^{-1} \cdot \underline{b}$$
$$I\underline{x} = A^{-1} \cdot \underline{b}$$
$$\underline{x} = A^{-1} \cdot \underline{b}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1/2 & 1/2 \end{pmatrix} \cdot \begin{pmatrix} 7 \\ 13 \end{pmatrix} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

Transpose of a matrix:

A (m×n) ⇝ A^T transpose of A, n×m-matrix

Ex: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 2 & 4 \\ 7 & -1 & 0 \\ 3 & 0 & 1 \end{pmatrix}$ $A^T = \begin{pmatrix} 1 & 7 & 3 \\ 2 & -1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$

($A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 7 & 0 \end{pmatrix}$ $A^T = \begin{pmatrix} 1 & 4 \\ 2 & 7 \\ 3 & 0 \end{pmatrix}$)

Defn: An n×n-matrix is <u>symmetric</u> if $A^T = A$

Ex: $A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & -1 & 4 \\ 3 & 4 & 7 \end{pmatrix}$

Also, let

$$\mathbf{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \text{and} \quad \mathbf{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix}.$$

Both $\mathbf{x}$ and $\mathbf{b}$ are matrices, called column matrices. The $n \times 1$ matrix $\mathbf{x}$ contains variables, and the $k \times 1$ matrix $\mathbf{b}$ contains the parameters from the right-hand side of the system. Then, the system of equations can be written as

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & a_{ij} & \vdots \\ a_{k1} & \cdots & a_{kn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ \vdots \\ b_k \end{pmatrix},$$

or simply as

$$A\mathbf{x} = \mathbf{b},$$

where $A\mathbf{x}$ refers to the matrix product of the $k \times n$ matrix A with the $n \times 1$ matrix $\mathbf{x}$. This product is a $k \times 1$ matrix, which must be made equal to the $k \times 1$ matrix $\mathbf{b}$. Check that carrying out the matrix multiplication in $A\mathbf{x} = \mathbf{b}$ and applying the definition of equality of matrices gives back exactly the original system of linear equations. The matrix notation is much more compact than writing out arrays of coefficients, and, as we shall see, it suggests how to find the solution to the system by analogy with the one-variable case.

EXERCISES

8.1 Let

$$A = \begin{pmatrix} 2 & 3 & 1 \\ 0 & -1 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 1 & -1 \\ 4 & -1 & 2 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix},$$

$$D = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad \text{and} \quad E = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

a) Compute each of the following matrices if it is defined:

$$\begin{array}{llllll} A+B, & A-D, & 3B, & DC, & B^T, & A^TC^T, \\ C+D, & B-A & AB, & CE, & -D, & (CE)^T, \\ B+C, & D-C, & CA, & EC, & (CA)^T, & E^TC^T. \end{array}$$

b) Verify that $(DA)^T = A^TD^T$.

c) Verify that $CD \neq DC$.

8.2 Check that

$$\begin{pmatrix} 2 & 3 & 1 & 4 \\ 0 & -1 & 2 & 1 \\ 5 & 0 & 6 & 0 \end{pmatrix} \begin{pmatrix} 2 & 3 \\ -1 & 0 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 11 \\ 2 & 3 \\ 10 & 21 \end{pmatrix}.$$

Note that the reverse product is not defined.

8.3 Show that if AB is defined, then B^TA^T is defined but A^TB^T need not be defined.

8.4 If you choose four numbers at random for the entries of a 2×2 matrix A, and four others for another 2×2 matrix B, AB will probably not equal BA. Carry out this procedure a few times.

8.5 It sometimes happens that $AB = BA$.

a) Check this for $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ and $B = \begin{pmatrix} 3 & -4 \\ -4 & 3 \end{pmatrix}$.

b) Show that if B is a scalar multiple of the 2×2 identity matrix, then $AB = BA$ for *all* 2×2 matrices A.

8.2 SPECIAL KINDS OF MATRICES

Special problems use special kinds of matrices. In this section we describe some of the important classes of $k \times n$ matrices which arise in economic analysis.

Square Matrix. $k = n$, that is, equal number of rows and columns.

Column Matrix. $n = 1$, that is, one column. For example,

$$\begin{pmatrix} a \\ b \\ c \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Row Matrix. $k = 1$, that is, one row. For example,

$$(2 \quad 1 \quad 0) \quad \text{and} \quad (2 \quad 3).$$

Diagonal Matrix. $k = n$ and $a_{ij} = 0$ for $i \neq j$, that is, a square matrix in which all nondiagonal entries are 0. For example,

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

> **Theorem 8.12** Any matrix A can be written as a product
>
> $$A = F_1 \cdots F_m \cdot U$$
>
> where the F_i's are elementary matrices and U is in reduced row echelon form. When A is nonsingular, $U = I$ and $A = F_1 \cdots F_m$.

EXERCISES

8.15 Check that

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ -1 & 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} .5 & 0 & -.5 \\ .5 & 0 & .5 \\ -.5 & 1 & -.5 \end{pmatrix}.$$

8.16 Verify that matrix (4) is the inverse of matrix (3) by direct matrix multiplication.

8.17 Suppose that $a = 0$ but $c \neq 0$ in (5). Show that one obtains the same inverse (7) for A.

8.18 Show by simple matrix multiplication that, if $ad - bc \neq 0$,

$$\frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

is both a left and a right inverse of A.

8.19 Use the technique of Example 8.3 to either invert each of the following matrices or prove that it is singular:

$$a)\ \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}, \quad b)\ \begin{pmatrix} 4 & 5 \\ 2 & 4 \end{pmatrix}, \quad c)\ \begin{pmatrix} 2 & 1 \\ -4 & -2 \end{pmatrix},$$

$$d)\ \begin{pmatrix} 2 & 4 & 0 \\ 4 & 6 & 3 \\ -6 & -10 & 0 \end{pmatrix}, \quad e)\ \begin{pmatrix} 2 & 1 & 0 \\ 6 & 2 & 6 \\ -4 & -3 & 9 \end{pmatrix},$$

$$f)\ \begin{pmatrix} 2 & 6 & 0 & 5 \\ 6 & 21 & 8 & 17 \\ 4 & 12 & -4 & 13 \\ 0 & -3 & -12 & 2 \end{pmatrix}.$$

8.20 Invert the coefficient matrix to solve the following systems of equations:

$$a)\ \begin{aligned} 2x_1 + x_2 &= 5 \\ x_1 + x_2 &= 3; \end{aligned} \qquad b)\ \begin{aligned} 2x_1 + x_2 \qquad\quad &= 4 \\ 6x_1 + 2x_2 + 6x_3 &= 20 \\ -4x_1 - 3x_2 + 9x_3 &= 3; \end{aligned}$$

$$c)\quad \begin{aligned} 2x_1 + 4x_2 \qquad &= 2 \\ 4x_1 + 6x_2 + 3x_3 &= 1 \\ -6x_1 - 10x_2 \qquad &= -6. \end{aligned}$$

8.21 Show that if A is $n \times n$ and $AB = BA$, then B is also $n \times n$.

8.22 For $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$, compute A^3, A^4, and A^{-2}.

8.23 Verify the statements about the inverses of elementary matrices in the last sentence of Example 8.5.

8.24 *a)* Use Theorem 8.8 to prove that a 2×2 lower- or upper-triangular matrix is invertible if and only if each diagonal entry is nonzero.

b) Show that the inverse of a 2×2 lower triangular matrix is lower triangular.

c) Show that the inverse of a 2×2 upper triangular matrix is upper triangular.

8.25 *a)* Prove Theorem 8.10.

b) Generalize part *c* to the case of the product of k nonsingular matrices.

c) Show by example that if A and B are invertible, $A + B$ need not be invertible.

d) Show that, when it exists, $(A + B)^{-1}$ is generally not $A^{-1} + B^{-1}$.

8.26 Prove Theorem 8.11.

8.27 *a)* Prove that $(AB)^k = A^k B^k$ if $AB = BA$.

b) Show that $(AB)^k \neq A^k B^k$ in general.

c) Conclude that $(A + B)^2$ does not equal $A^2 + 2AB + B^2$ unless $AB = BA$.

8.28 What is the inverse of the $n \times n$ diagonal matrix

$$D = \begin{pmatrix} d_1 & 0 & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & 0 & 0 & \cdots & d_n \end{pmatrix}?$$

8.29 Show that the inverse of a 2×2 symmetric matrix S is symmetric.

8.30 Show that the inverse of an $n \times n$ upper-triangular matrix U is upper-triangular. Can you find an easy argument to extend this result to lower-triangular matrices? [Hint: There are a number of ways to do the first part. You can use the inversion method described in the proof of Theorem 8.7, keeping track of the status of the 0s below the diagonal. Or, you can show by direct calculation that $BU = I$ implies that B has only 0s below the diagonal.]

8.31 Show that for any permutation matrix P, $P^{-1} = P^T$.

8.32 Use Gauss-Jordan elimination to derive a criterion for the invertibility of 3×3 matrices similar to the $ad - bc$ criterion for the 2×2 case. For simplicity, assume that no row interchanges are needed in the elimination process.

8.33 The definitions of left inverse and right inverse apply to nonsquare matrices. Use the ideas in the proof of Theorem 8.7 to prove the following statements for an $m \times n$ matrix A, where $m \neq n$.

a) A nonsquare matrix cannot have both a left and a right inverse.

b) If A has one left (right) inverse, it has infinitely many.

c) If $m < n$, A has a right inverse if and only if $\operatorname{rank} A = m$.

d) If $m > n$, A has a left inverse if and only if $\operatorname{rank} A = n$.

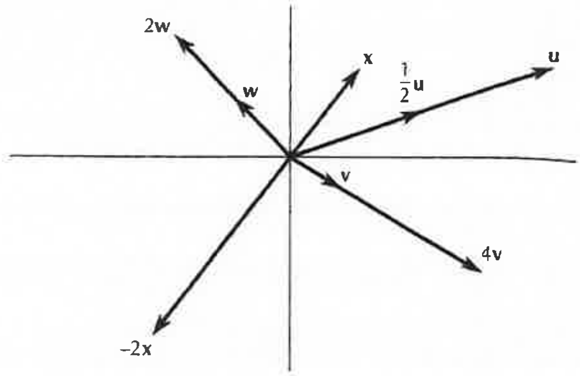


Figure 10.11 *Scalar multiplication in the plane.*

There are distributive laws in Euclidean spaces as well. It is easy to see that vector addition distributes over scalar multiplication and that scalar multiplication distributes over vector addition:

(*a*) $(r + s)\mathbf{u} = r\mathbf{u} + s\mathbf{u}$ for all scalars r, s and vectors $\mathbf{u}$.
(*b*) $r(\mathbf{u} + \mathbf{v}) = r\mathbf{u} + r\mathbf{v}$ for all scalars r and vectors $\mathbf{u}$, $\mathbf{v}$.

Any set of objects with a vector addition and scalar multiplication which satisfies the rules we have outlined in this section is called a **vector space**. The elements of the set are called **vectors**. (The operations of vector addition and scalar multiplication are the operations of matrix addition and scalar multiplication of matrices, respectively, applied to $1 \times n$ or $n \times 1$ matrices, as defined in Section 1 of Chapter 8. The scalar product of the next section will also correspond to a matrix operation.)

EXERCISES

10.5 Let $\mathbf{u} = (1, 2)$, $\mathbf{v} = (0, 1)$, $\mathbf{w} = (1, -3)$, $\mathbf{x} = (1, 2, 0)$, and $\mathbf{z} = (0, 1, 1)$. Compute the following vectors, whenever they are defined: $\mathbf{u} + \mathbf{v}$, $-4\mathbf{w}$, $\mathbf{u} + \mathbf{z}$, $3\mathbf{z}$, $2\mathbf{v}$, $\mathbf{u} + 2\mathbf{x}$, $\mathbf{u} - \mathbf{v}$, $3\mathbf{x} + \mathbf{z}$, $-2\mathbf{x}$, $\mathbf{w} + 2\mathbf{x}$.

10.6 Carry out all of the possible operations in Exercise 10.5 *geometrically*.

10.7 Show that $-\mathbf{u} = (-1)\mathbf{u}$.

10.8 Prove the distributive laws for vectors in $\mathbf{R}^n$.

10.9 Use Figure 10.12 to give a geometric proof of the **associative law** for vector addition: $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$.