Plan

1. Analysis of functions and graphs using the derivative
2. Integration

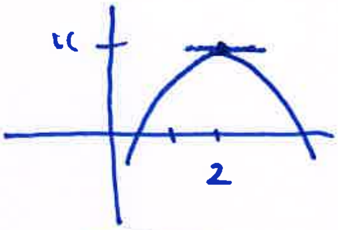
Ex: $f(x) = -x^2+4x+7$

$f'(x) = -2x+4 = 0$

$\underline{x=2}$ Stat. pt.

$f''(x) = -2 < 0$

$\cap$ f is concave



x=2 is a global max pt

Defn: A $\underline{\text{stationary pt.}}$ for f is a pt. with $f'(x)=0$.

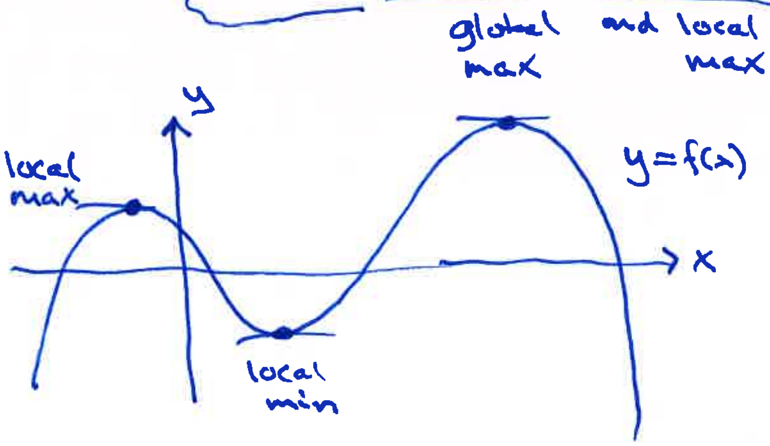
Defn: A pt. $x=x^*$ is called

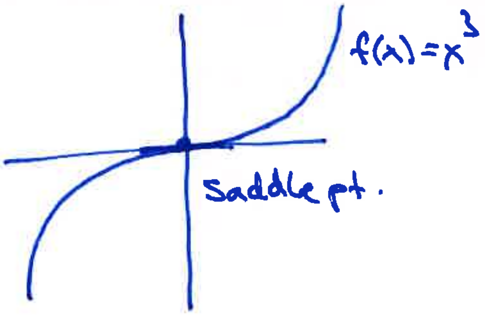
global max / max : $f(x^*) \geq f(x)$ for all x

local max : $f(x^*) \geq f(x)$ for all x close to $\underline{x^*}$

global min / min : $f(x^*) \leq f(x)$ for all x

local min : $f(x^*) \leq f(x)$ for all x close to x^*





Results:

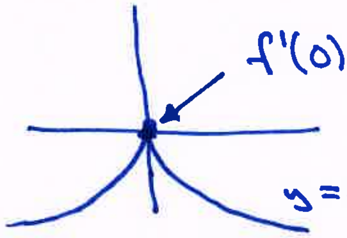
If $x=x^*$ is a max/min for f, then one of the following holds:

i) $x=x^*$ is stationary pt, i.e. $f'(x^*)=0$

ii) $f'(x^*)$ does not exist

iii) $x=x^*$ is a boundary pt.

Ex:



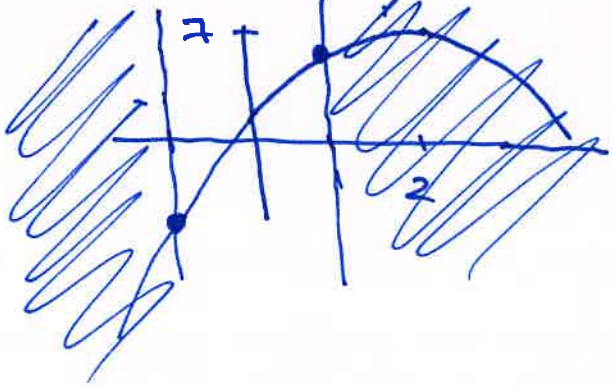
$$y=f(x) = -\sqrt[3]{x^2}$$
$$= -x^{2/3}$$

$f(0)=0$ $f'(0)$ does not exist

$x=0$ is max

$$f'(x) = -\frac{2}{3}x^{-1/3}$$
$$= -\frac{2}{3}\cdot\frac{1}{\sqrt[3]{x}}$$
$$= -\frac{2}{3\cdot\sqrt[3]{x}}$$

Ex: $f(x) = -x^2+4x+3$, $-1 \leq x \leq 1$



$$-x^2+4x+3$$
$$=(-x^2+4x-4)+7$$
$$=-(x-2)^2+7$$

$x=-1$ is min
$x=1$ is max } boundary pts.

Results:

① If f is defined on a closed intervall $[a,b]$ then f has a max or a min.

② If f is convex, then any stationary pt is global min
—"— concave, then —"— is global max.

$(e^{u(x)})' = e^{u(x)} \cdot u'(x)$

Ex: $f(x) = 4e^{2x} + 3e^{-3x}$

$f'(x) = 4\cdot(e^{2x}\cdot 2) + 3\cdot(e^{-3x}\cdot(-3))$

$= 8e^{2x} - 9e^{-3x}$

$= 8\cdot e^{2x} - 9\cdot\frac{1}{e^{3x}}$

$= \frac{8\cdot e^{2x}\cdot e^{3x}}{e^{3x}} - \frac{9}{e^{3x}} = \frac{8e^{5x}-9}{e^{3x}}$

Stat. pts: $f' = \boxed{e^{-3x}(8e^{5x}-9)} = 0$

$\cancel{e^{-3x} = 0}$ or $8e^{5x} - 9 = 0$

$\frac{8e^{5x}}{8} = \frac{9}{8}$

$e^{5x} = 9/8$

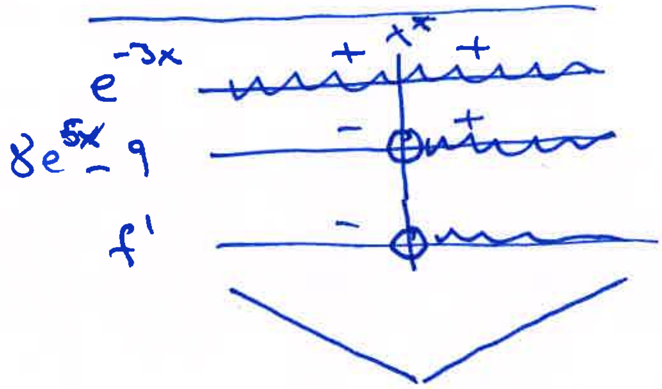
$\ln(e^{5x}) = \ln(9/8)$

$5x = \ln(9/8)$

$x = \frac{\ln(9/8)}{5}$

Stat. pts: $x^* = \frac{1}{5}\ln(9/8)$

Sign diagram for f':



Conclusion:

$x = x^* = \frac{1}{5}\ln(9/8)$

is a global min for f.

There is no global max.

Second derivative test:

If $x = x^*$ is a stationary pt for f, then:

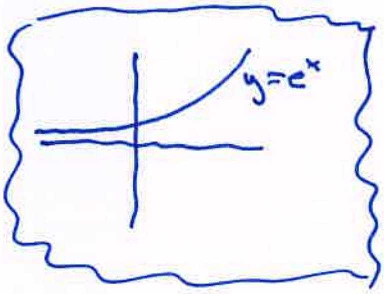
$f''(x^*) > 0 \Longrightarrow x = x^*$ is a <u>local min</u>

$f''(x^*) < 0 \Longrightarrow x = x^*$ is a <u>local max</u>

<u>Ex</u>: $f'(x) = 8e^{2x} - 9e^{-3x}$

$$f''(x) = 8 \cdot e^{2x} \cdot 2 - 9 \cdot e^{-3x} \cdot (-3)$$

$$= 16\underline{e^{2x}} + 27\underline{e^{-3x}} > 0 \quad \text{for all } x.$$



f <u>convex</u> $\underset{\text{convex opt.}}{\Longrightarrow} x = \frac{1}{5}\ln(9/8)$ global min

$f''(\frac{1}{5}\ln(9/8)) > 0 \underset{\text{second derivative test}}{\Longrightarrow} x = \frac{1}{5}\ln(9/8)$ <u>local min</u>

<u>Ex</u>:

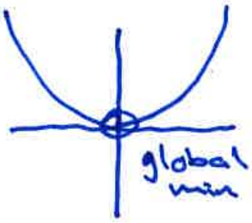
$f(x) = x^4$

$f'(x) = 4x^3 = 0$

$\underline{x = 0}$

$f''(x) = 12x^2$

$\underline{f''(0) = 0}$



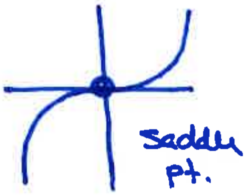
$f(x) = x^3$

$f' = 3x^2 = 0$

$\underline{x = 0}$

$f'' = 6x$

$\underline{f''(0) = 0}$



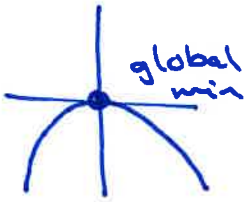
$f(x) = -x^4$

$f' = -4x^3 = 0$

$\underline{x = 0}$

$f'' = -12x^2$

$\underline{f''(0) = 0}$



Ex: $f(x) = \frac{3}{5}\ln(1+x) + \frac{2}{5}\ln(1-x)$, $0 \le x < 1$

$= 0.6\ln(1+x) + 0.4\ln(1-x)$ $D_f = [0, 1)$

$$f'(x) = \frac{3}{5}\cdot\frac{1}{1+x} + \frac{2}{5}\cdot\frac{-1}{1-x}$$

$$= \frac{3/5(1-x)}{(1+x)(1-x)} \quad \frac{2/5(1+x)}{(1-x)(1+x)}$$

$$= \frac{3/5(1-x) - 2/5(1+x)}{(1+x)(1-x)} \begin{matrix}\cdot 5\\ \cdot 5\end{matrix}$$

$$= \frac{3-3x-2-2x}{5(1+x)(1-x)} = \frac{1-5x}{5(1+x)(1-x)} = 0$$

$$\boxed{\begin{aligned}&\left(\ln(u(x))\right)'\\ &= \frac{1}{u(x)}\cdot u'(x)\\ &= \frac{u'(x)}{u(x)}\end{aligned}}$$

$1-5x=0$

$\underline{x = 1/5}$

(in D_f)

Stationary pts:

$\underline{\underline{x^* = 1/5 = 0.2}}$

$$f''(x) = \left(\frac{3/5}{1+x} - \frac{2/5}{1-x}\right)' =$$

$$= \frac{3}{5}\left(-\frac{1}{(1+x)^2}\right) - \frac{2}{5}\left(-\frac{-1}{(1-x)^2}\right)$$

$$= -\frac{3}{5}\cdot\frac{1}{(1+x)^2} - \frac{2}{5}\,\frac{1}{(1-x)^2} < 0$$

for all x

$$\boxed{\begin{aligned}&\left(\frac{1}{u(x)}\right)' = (u(x)^{-1})'\\ &= -1\cdot u(x)^{-2}\cdot u'(x)\\ &= -\frac{u'(x)}{u(x)^2}\end{aligned}}$$

$\Downarrow$

f concave $\Rightarrow$ $x^* = \underline{1/5 = 0.2}$

$\frown$ is global max.

② Integration

$f(x)$: any function in one variable

Defn: $F(x)$ is an antiderivative of f if $F'(x) = f(x)$.

Ex: $f(x) = 2x$

$F(x) = x^2+1$ is an antiderivative

$F(x) = \underline{x^2+C}$ is the general antiderivative of f

$$\boxed{\text{Indefinite integral: } \int f(x)\,\underline{\underline{dx}} = F(x) + C \quad \text{if } F'(x) = f(x)}$$

Ex: $\int 2x\,dx = \underline{\underline{x^2+C}}$

Integration rules:

i) Power rule: $\int x^n\,dx = \frac{1}{n+1}\cdot x^{n+1} + C$, for $n \neq -1$

$$\left(\frac{1}{n+1}x^{n+1}\right)' = \cancel{\frac{1}{n+1}}\cdot\cancel{(n+1)}x^n = x^n$$

ii) $\int \frac{1}{x}\,dx = \ln|x| + C$

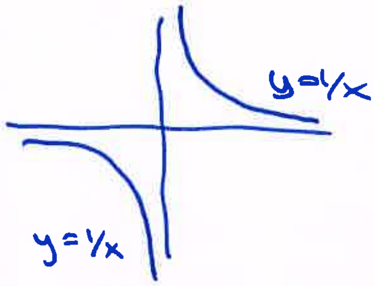
iii) $\int e^x\,dx = e^x + C$

iv) $\int u(x) \pm v(x)\,dx = \int u(x)\,dx \pm \int v(x)\,dx$

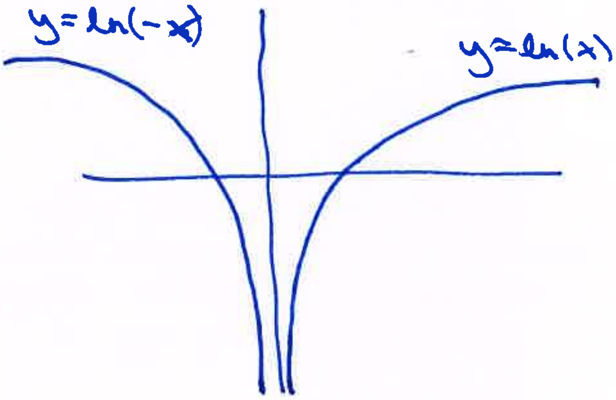
v) $\int c\cdot u(x)\,dx = c\cdot\int u(x)\,dx$ when c is a constant

Why is $\int \frac{1}{x}\,dx = \ln|x| + C$?

$f(x) = 1/x$



$F(x) = \ln(x)$, $x > 0$



$$(\ln(-x))' = \frac{1}{-x} \cdot (-1) = 1/x$$

General antiderivative:

$$F(x) = \begin{cases} \ln(x) , & x > 0 \\ \ln(-x), & x < 0 \end{cases}$$

$$= \ln|x|$$

Ex: $\int x^2 - 3x + 2\,dx = \int x^2\,dx - \int 3x\,dx + \int 2\,dx$

$$= \underset{+C_1}{\frac{1}{3}x^3} - 3\Big(\underset{+C_2}{\frac{1}{2}x^2}\Big) + \underset{+C_3}{2x} = \underline{\underline{\frac{1}{3}x^3 - \frac{3}{2}x^2 + 2x + C}}$$

Integration techniques:

1) Integration by parts (prod.)
2) Substitution (composite fn.)
3) Partial fractions (rational fn.)

1) Integration by parts:

$$\boxed{\begin{array}{c}\int u'\cdot v\,dx = u\cdot v - \int u\cdot v'\,dx \\ u=u(x),\ v=v(x)\end{array}}$$

$$(u\cdot v)' = u'v + u\cdot v'$$

$$\int (u\cdot v)'\,dx = \int u'v\,dx + \int uv'\,dx$$

$$uv = \int u'v\,dx + \int uv'\,dx$$

$$\int x\cdot e^x\,dx$$

$$\boxed{\begin{array}{ll} u = e^x & v = x \\ u' = e^x & v' = 1 \end{array}}$$

$$= xe^x - \int e^x\,dx$$

$$= \underline{\underline{xe^x - e^x + C}}$$

$$\int x\cdot \ln(x)\,dx = \frac{1}{2}x^2\cdot \ln(x) - \int \frac{1}{2}x^2\cdot\frac{1}{x}\,dx$$

$$\boxed{\begin{array}{ll} u = \frac{1}{2}x^2 & v = \ln x \\ u' = x & v' = 1/x \end{array}}$$

$$= \frac{1}{2}x^2\ln(x) - \int \frac{1}{2}x\,dx$$

$$= \frac{1}{2}x^2\ln(x) - \frac{1}{2}\cdot\frac{1}{2}x^2 + C$$

$$= \underline{\underline{\frac{1}{2}x^2\cdot\ln(x) - \frac{1}{4}x^2 + C}}$$

$$\int \ln(x)\,dx = \int 1\cdot\ln(x)\,dx = x\cdot\ln(x) - \int x\cdot\frac{1}{x}\,dx$$

$$\boxed{\begin{array}{ll} u = x & v = \ln(x) \\ u' = 1 & v' = 1/x \end{array}}$$

$$= x\ln(x) - \int 1\,dx$$

$$= \underline{\underline{x\ln(x) - x + C}}$$

2) Substitution: $\frac{1}{1+x} = \frac{1}{u}, \quad u = 1+x$

$$\int \frac{1}{1+x} \boxed{dx} = \int \frac{1}{u}\,du = \ln|u| + C$$

$$\boxed{\begin{aligned} u &= 1+x \\ \boxed{du} &\boxed{= u' \cdot dx} \\ \Rightarrow du &= dx \end{aligned}}$$

$$= \underline{\ln|1+x| + C}$$

$$\int e^{2x}\,dx = \int e^{u} \cdot \frac{1}{2}\,du$$

$$\boxed{\begin{aligned} u &= 2x \\ du &= 2\,dx \end{aligned}} \qquad dx = \frac{1}{2}\,du$$

$$= \int \frac{1}{2} e^{u}\,du = \frac{1}{2} \cdot e^{u} + C = \underline{\frac{1}{2} e^{2x} + C}$$

Check: $\left(\frac{1}{2} e^{2x}\right)' = \frac{1}{\cancel{2}} \cdot e^{2x} \cdot \cancel{2} = e^{2x}$

Ex: $\int \frac{x}{x^2+1} \boxed{dx} = \int \frac{\cancel{x}}{u} \cdot \frac{1}{2\cancel{x}}\,du$

$$\boxed{\begin{aligned} u &= x^2+1 \\ du &= 2x \cdot dx \end{aligned}} \qquad dx = \frac{1}{2x}\,du$$

$$= \int \frac{1}{2} \cdot \frac{1}{u}\,du = \frac{1}{2} \cdot \ln|u| + C$$

$$= \frac{1}{2} \underline{\ln(x^2+1) + C}$$

3) Partial fractions: Postponed for Lecture 6

Problems for Lecture 4

1. Compute the integrals

a) $\int x^2 \ln x \, dx$

b) $\int \frac{x^3 + 2x^2 + 1}{x} \, dx$

c) $\int x e^{-x^2} dx$

d) $\int \frac{x}{x^2+1} \, dx$

e) $\int \frac{1}{(2x-3)^2} \, dx$

f) $\int x^3 e^{-x^2} dx$

g) $\int e^{\sqrt{x}} dx$

2 Simplify the expressions using polynomial division:

a) $\frac{x^3}{x^2-1}$

b) $\frac{x^2+2x-3}{x+1}$

c) $\frac{x^3+x^2+x+1}{x+1}$

d) $\frac{x^4+1}{x-1}$

3. Compute the integrals:

a) $\int \frac{3}{2x-4} \, dx$

b) $\int \frac{1}{x^2+x} \, dx$

c) $\int \frac{x}{x^2-4x-5} \, dx$

d) $\int \frac{x^2}{x^2-1} \, dx$

e) $\int \frac{1}{x^2-4x+4} \, dx$

Solutions for Lecture 4

1. a) $\int \underset{\substack{\shortparallel \\ v'}}{x^2} \underset{\substack{\shortparallel \\ u}}{\ln x}\, dx = \underset{\substack{\shortparallel \\ v}}{\tfrac{1}{3}x^3} \cdot \underset{\substack{\shortparallel \\ u}}{\ln x} - \int \underset{\substack{\shortparallel \\ v}}{\tfrac{1}{3}x^3} \cdot \underset{\substack{\shortparallel \\ u'}}{\tfrac{1}{x}}\, dx$

$= \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 dx = \underline{\frac{1}{3}x^3 \ln x - \frac{1}{9}x^3 + C}$

b) $\int \frac{x^3+2x^2+1}{x} dx = \int x^2 + 2x + \frac{1}{x}\, dx = \underline{\frac{1}{3}x^3 + x^2 + \ln|x| + C}$

c) $\int x e^{-x^2} dx = \int \cancel{x} e^u \cdot \frac{du}{-2\cancel{x}} = -\frac{1}{2}\int e^u du = -\frac{1}{2}e^u + C$

$u = -x^2$
$du = -2x\,dx$

$= \underline{-\frac{1}{2} e^{-x^2} + C}$

d) $\int \frac{x}{x^2+1} dx = \int \frac{\cancel{x}}{u} \cdot \frac{du}{2\cancel{x}} = \frac{1}{2}\int \frac{1}{u} du = \frac{1}{2}\ln|u| + C$

$u = x^2+1$
$du = 2x\,dx$

$= \frac{1}{2}\ln|x^2+1| + C$

$= \underline{\frac{1}{2}\ln(x^2+1) + C}$

e) $\int \frac{1}{(2x-3)^2} dx = \int \frac{1}{u^2}\frac{du}{2} = \frac{1}{2}\int u^{-2} du$

$u = 2x-3$
$du = 2\,dx$

$= \frac{1}{2}\left(-\frac{1}{u}\right) + C$

$= \underline{-\frac{1}{2} \cdot \frac{1}{2x-3} + C}$

f) $\int x^3 e^{-x^2} dx = \int x^3 e^u \frac{du}{-2x} = \int -\frac{1}{2}x^2 e^u du$

$u = -x^2$
$du = -2x\,dx$

$= -\frac{1}{2}\int (-u) e^u du$

$x^2 = -u$

$= \frac{1}{2}\int u e^u du \underset{\substack{\uparrow \\ \text{int. by parts}}}{=} u e^u - \int 1 \cdot e^u du = u e^u - e^u + C$

$= \underline{-x^2 e^{-x^2} - e^{-x^2} + C}$

g) $\int e^{\sqrt{x}}\,dx = \int e^{u} \cdot 2\sqrt{x}\,du$

$u=\sqrt{x}$
$du = \frac{1}{2\sqrt{x}}\,dx$

$= \int e^{u} \cdot 2u\,du$

$\sqrt{x} = u$

$= \int 2ue^{u}\,du = 2ue^{u} - \int 2e^{u}\,du$

int. by parts

$= 2ue^{u} - 2e^{u} + C = \underline{2\sqrt{x}\,e^{\sqrt{x}} - 2e^{\sqrt{x}} + C}$

<u>2.</u> a) $x^{3} : x^{2}-1 = x \Rightarrow \underline{\frac{x^{3}}{x^{2}-1} = x + \frac{x}{x^{2}-1}}$

$-(x^{3}-x)$

$\overline{\quad x \quad}$

b) $x^{2}+2x-3 : x+1 = x+1 \Rightarrow \frac{x^{2}+2x-3}{x+1} = \underline{x+1+\frac{-4}{x+1}}$

$-(x^{2}+x\quad)$

$\overline{\quad x-3 \quad}$

$-(x+1)$

$\overline{\quad -4 \quad}$

c) $x^{3}+x^{2}+x+1 : x+1 = x^{2}+1 \Rightarrow \frac{x^{3}+x^{2}+x+1}{x+1} = \underline{x^{2}+1}$

$-(x^{3}+x^{2}\quad)$

$\overline{\quad x+1 \quad}$

$x+1$

$\overline{\quad 0 \quad}$

d) $x^{4}+1 : x-1 = x^{3}+x^{2}+x+1 \Rightarrow \frac{x^{4}+1}{x-1} = \underline{x^{3}+x^{2}+x+1+\frac{2}{x-1}}$

$-(x^{4}-x^{3})$

$\overline{\quad x^{3}+1 \quad}$

$-(x^{3}-x^{2})$

$\overline{\quad x^{2}+1 \quad}$

$-(x^{2}-x)$

$\overline{\quad x+1 \quad}$

$-(x-1)$

$\overline{\quad 2 \quad}$

3. a) $\int \frac{3}{2x-4}\,dx = \int \frac{3}{u}\frac{du}{2} = \frac{3}{2}\int \frac{1}{u}\,du$

$u = 2x-4$
$du = 2dx$

$= \frac{3}{2}\ln|2x-4| + C$

b) $\int \frac{1}{x^2+x}\,dx \underset{\uparrow}{=} \int \frac{1}{x} + \frac{-1}{x+1}\,dx = \ln|x| - \ln|x+1| + C$

$\frac{1}{x^2+x} = \frac{1}{x\cdot(x+1)} = \frac{A}{x} + \frac{B}{x+1} \quad \Big|\cdot(x+1)x$

$1 = A\cdot(x+1) + Bx$

$= \underset{0}{\underset{\shortparallel}{(A+B)}}x + \underset{1}{\underset{\shortparallel}{(A)}}$

$A = 1, \quad B = -1$

$\underset{\uparrow}{=} \ln\left|\frac{x}{x+1}\right| + C$

rules for logarithms

c) $\int \frac{x}{x^2-4x-5}\,dx \underset{\uparrow}{=} \int \frac{5/6}{x-5} + \frac{1/6}{x+1}\,dx$

$x^2-4x-5 = (x-5)(x+1)$

$\frac{x}{x^2-4x-5} = \frac{A}{x-5} + \frac{B}{x+1} \quad \Big|\cdot(x-5)(x+1)$

$x = A(x+1) + B(x-5)$

$= \underset{1}{\underset{\shortparallel}{(A+B)}}x + \underset{0}{\underset{\shortparallel}{(A-5B)}}$

$A+B = 1$
$A-5B = 0 \Rightarrow A = 5B$

$5B + B = 1$

$6B = 1$

$A = 5/6 \quad B = 1/6$

$= \frac{5}{6}\ln|x-5| + \frac{1}{6}\ln|x+1| + C$

$= \frac{1}{6}\left(\ln|x-5|^5 + \ln|x+1|\right) + C$

$= \frac{1}{6}\ln|(x-5)^5(x+1)| + C$

d) $\int \frac{x^2}{x^2-1}\,dx = \int 1 + \frac{1}{x^2-1}\,dx = x + \int \frac{1}{(x-1)(x+1)}\,dx$

$$\begin{array}{l} x^2 : x^2-1 = 1 \\ -(x^2-1) \\ \hline 1 \end{array}$$

$\frac{1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1} \quad | \cdot (x-1)(x+1)$

$1 = A(x+1) + B(x-1)$

$= \underset{\overset{\shortparallel}{0}}{(A+B)}x + \underset{\overset{\shortparallel}{1}}{(A-B)}$

$$\begin{array}{l} A+B=0 \\ A-B=1 \\ \hline 2A \;\; = 1 \end{array} \quad \longrightarrow \underline{A = 1/2}, \; B = \underline{-1/2}$$

$= x + \int \frac{1/2}{x-1} + \frac{-1/2}{x+1}\,dx = x + \frac{1}{2}\ln|x-1| - \frac{1}{2}\ln|x+1| + C$

$= x + \frac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C$

e) $\int \frac{1}{x^2-4x+4}\,dx = \int \frac{1}{(x-2)^2}\,dx = \int \frac{1}{u^2}\,du = -\frac{1}{u} + C$

$u = x-2$
$du = 1 \cdot dx$

$= -\frac{1}{x-2} + C$