Plan

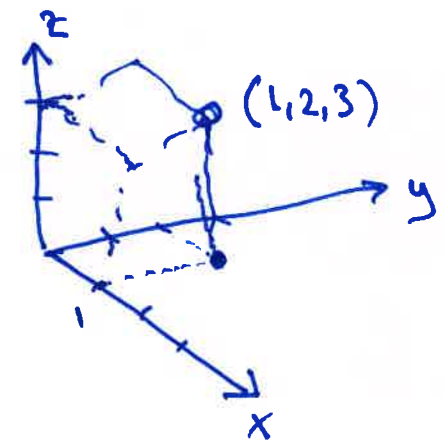
1 Functions in two variables

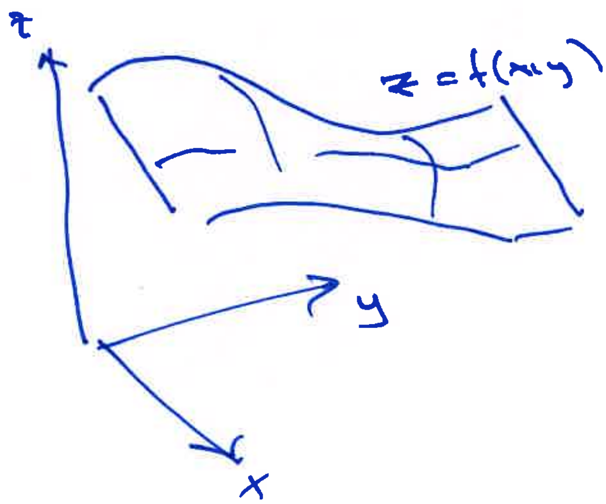
$f(x,y)$ = some expression in x and y

Ex: $f(x,y) = \overbrace{x^3 - xy + y^2}^{\text{functional expr. of } f}$, $D_f = \mathbb{R}^2$ = all (x,y) with x,y in $\mathbb{R}$

$x=1,\ y=2$: $f(1,2) = 1^3 - 1\cdot 2 + 2^2$
$= \underline{3}$ $z=3$

Graph: $z = f(x,y)$





② Partial derivatives

Ex: $f(x,y) = x^3 - xy + y^2$ — think of y as a const

$f'_x(x,y) = 3x^2 - y\cdot 1 + 0 = \underline{3x^2 - y}$

$f'_y(x,y) = 0 - x\cdot 1 + 2y = \underline{-x + 2y}$ — think of x as a const.

Defn:

$$\boxed{\begin{aligned} f'_x(x,y) &= \lim_{h\to 0} \frac{f(x+h,y) - f(x,y)}{h} \\ f'_y(x,y) &= \lim_{h\to 0} \frac{f(x,y+h) - f(x,y)}{h} \end{aligned}}$$

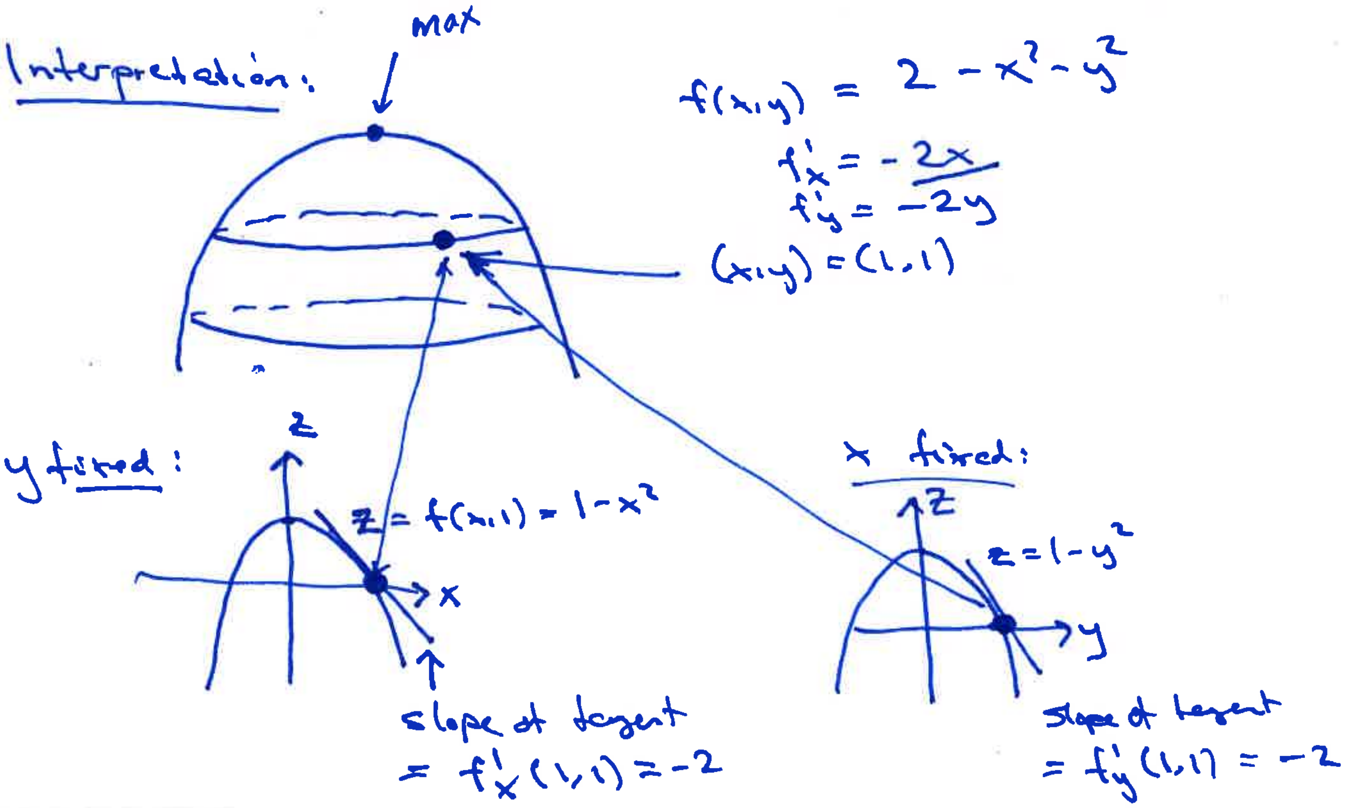
Interpretation:

$f(x,y) = 2 - x^2 - y^2$

$f'_x = -2x$

$f'_y = -2y$

$(x,y) = (1,1)$



Defn: A stationary pt. for f is a pt. (x,y) s.t.

$$\left.\begin{cases} f'_x = 0 \\ f'_y = 0 \end{cases}\right\}$$ FOC (first order conditions)

③ Unconstrained Optimization

max / min $f(x,y)$

Candidate pts = pts that can be max / min:

i) stationary pts for f

ii) pts where f'_x or f'_y are not defined

iii) boundary pts.

Ex: $f(x,y) = x^3 - xy + y^2$, $D_f = \mathbb{R}^2$

$$\left.\begin{array}{l} f'_x = 3x^2 - y = 0 \\ f'_y = -x + 2y = 0 \end{array}\right\} \text{FOC}$$

Solution = stat. pts.

$x = 2y \Rightarrow 3 \cdot (2y)^2 - y = 0$

$3 \cdot 4y^2 - y = 0$

$12y^2 - y = 0$

$y(12y - 1) = 0$

$y = 0$ or $y = 1/12$

$x = 0$ $\quad$ $x = 1/6$

Stat. pts: $(x,y) = (0,0)$, $(1/6, 1/12)$

Defn: (x^*,y^*) is a <u>global max</u> for f if $f(x^*,y^*) \geq f(x,y)$ for all (x,y).

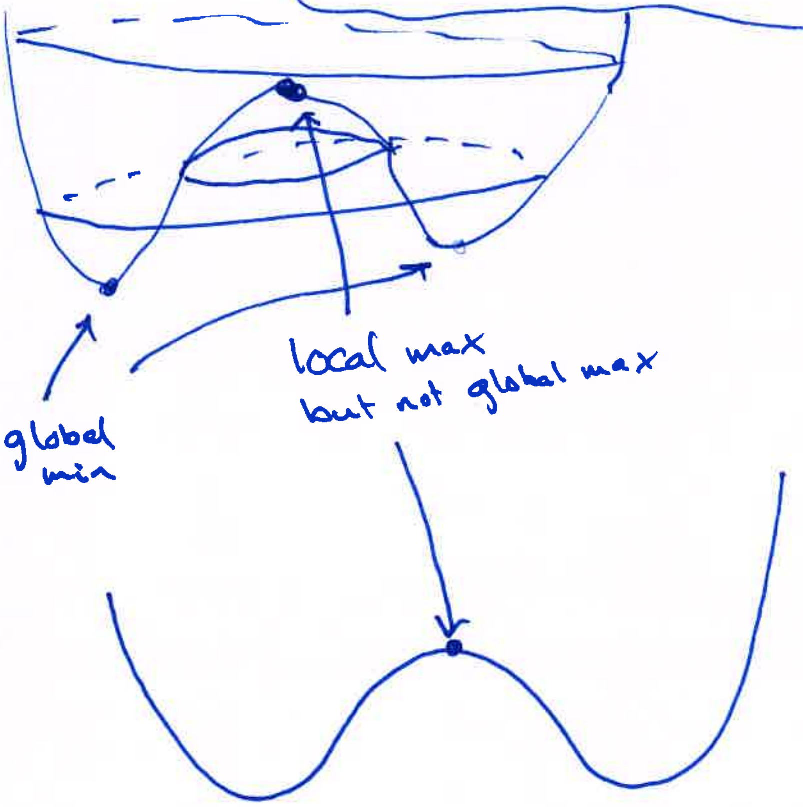
<u>max</u>

" is a <u>local max</u> for f if $f(x^*,y^*) \geq f(x,y)$ for all (x,y) close to (x^*,y^*)

" is a <u>global min</u> for f if $f(x^*,y^*) \leq f(x,y)$ for all (x,y)

<u>min</u>

" is a <u>local min</u> for f if $f(x^*,y^*) \leq f(x,y)$ for all (x,y) close to (x^*,y^*)



<u>Defn:</u>

A <u>saddle pt</u> is a stationary pt that is not local max, not local min.

Stationary pt. — local min / local max / saddle pt.

local classification of stat. pts.

Hessian matrix:

Ex: $f(x,y) = x^3 - xy + y^2$

first order der.: $f'_x = 3x^2 - y$, $f'_y = -x + 2y$

Stat. pts: $(x,y) = (0,0), (1/6, 1/12)$

$H(f)(1/6, 1/12) = \begin{pmatrix} 1 & -1 \\ -1 & 2 \end{pmatrix}$

$\det = 1 \cdot 2 - (-1)^2 = 1 > 0$

$A = 1 > 0$

$(1/6, 1/12)$ local min

$f''_{xx} = 6x \qquad f''_{xy} = -1$

$f''_{yx} = -1 \qquad f''_{yy} = 2$

$H(f)(x,y) = \begin{pmatrix} 6x & -1 \\ -1 & 2 \end{pmatrix} = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix}$

$H(f)(0,0) = \begin{pmatrix} 0 & -1 \\ -1 & 2 \end{pmatrix}$

$\det = 0 \cdot 2 - (-1)^2 = -1$

$(0,0)$ is saddle pt

Note: If f is a "nice" function, then $H(f)$ is symmetric, i.e. $f''_{xy} = f''_{yx}$

Second derivative test: (x^*, y^*) stationary pt. for f

Look at $H(f)(x^*, y^*) = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$:

$\det = AC - B^2 > 0$, $A > 0$: (x^*, y^*) local min.

$\det = AC - B^2 > 0$, $A < 0$: (x^*, y^*) local max

$\det = AC - B^2 < 0$: (x^*, y^*) saddle pt.

Explanation: $H(f)(x^*,y^*) = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$

det $= AC - B^2 > 0$:

$AC > B^2 \geq 0 \Rightarrow AC > 0$

- $A > 0, C > 0$
- $A < 0, C < 0$

$A < 0$

$C < 0$

$\to x$

$\to y$

local max

$A = f''_{xx}(x^*,y^*) > 0$

$C = f''_{yy}(x^*,y^*) > 0$

$\to x$

$\to y$

local min

$\text{tr } H(f)(x^*,y^*) = \text{tr} \begin{pmatrix} A & B \\ B & C \end{pmatrix} = A + C$

If det $= AC - B^2 > 0$:

- $A > 0, C > 0$, $A + C > 0$
- $A < 0, C < 0$, $A + C < 0$

Note: If det $H(f)(x^*,y^*) = AC - B^2 = 0$, then the second derivative test is inconclusive

Ex: $f(x,y) = x^3 - xy + y^2$

Stat. pts: $(0,0)$, $(1/6, 1/12)$

saddle pt. — local min

What about global max/min?

1) No global max

2) Candidate for global min: $(x,y) = (1/6, 1/12)$

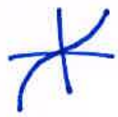
$$f(1/6, 1/12) = \left(\tfrac{1}{6}\right)^3 - \tfrac{1}{6}\cdot\tfrac{1}{12} + \left(\tfrac{1}{12}\right)^2$$

$$f(x,y) = x^3 - xy + y^2$$

$$f(x,0) = x^3$$

$$f(-10,0) = -1000$$

$$f(x,0) = x^3 \to -\infty \text{ when } x \to -\infty$$



No global min

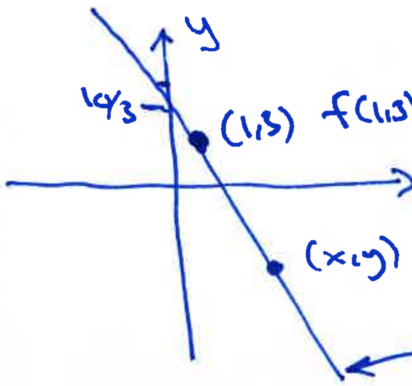
8 Lagrange problems (constrained optimization)

$$\boxed{\max/\min\ f(x,y) \quad \text{when} \quad g(x,y)=a}$$

Ex: $\max/\min\ \underbrace{f(x,y)=x^2+y^2}_{\text{fn. that we want to max/min.}}$ when $\underbrace{x+3y=10}_{\text{constraint}}$

$g(x,y)=x+3y$

$a=10$



$(1,3)$ $f(1,3)=\underline{10}$ = objective fn.

$(x,y) \rightsquigarrow f(x,y)=x^2+y^2$

$\leftarrow x+3y=10$

$x+3y=10$

$3y=-x+10$

$y=-\frac{1}{3}x+\frac{10}{3}$

Lagrange's method: method to find candidates for max/min in Lagrange problem

$(x,y;\lambda)=\underline{\underline{(1,3;2)}}$

$L=f(x,y)-\lambda\cdot g(x,y)$

$=x^2+y^2-\lambda\cdot(x+3y)$

λ: Lagrange multiplier

$$\left.\begin{aligned} L'_x &= 2x-\lambda\cdot 1=0 \\ L'_y &= 2y-\lambda\cdot 3=0 \end{aligned}\right\}\text{FOC}$$

$$x+3y=10\ \}\ C$$

Solution = Candidate pts.

1) $2x=\lambda \Rightarrow x=\lambda/2=\underline{1}$

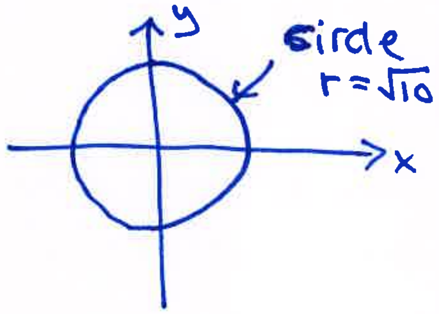
2) $2y=3\lambda \Rightarrow y=3\lambda/2=\underline{3}$

$x+3y=(\lambda/2)+3(3\lambda/2)=10 \quad |\cdot 2$

$\lambda + 9\lambda = 20$

$10\lambda=20 \quad \lambda=\underline{\underline{2}}$

Ex: max/min $f = x+3y$ when $x^2+y^2=10$



Extreme Value Thm:

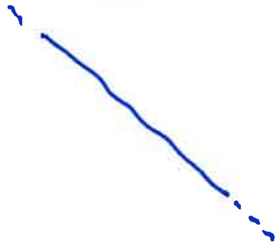
If f is continuous fn. on a closed and bounded set, then f has a max and a min.

$x^2+y^2=10$ is closed and bounded

closed: =

bounded: contained in a rectangle with finite sides

$x+3y=10$ is not bounded



Candidate pts:

$$L = f(x,y) - \lambda \cdot g(x,y)$$
$$= x+3y - \lambda\cdot(x^2+y^2)$$

$$\boxed{\begin{aligned} L'_x &= 1-\lambda\cdot 2x = 0 \\ L'_y &= 3-\lambda\cdot 2y = 0 \\ & x^2+y^2 = 10 \end{aligned}}$$

$\left.\begin{matrix} L'_x \\ L'_y \end{matrix}\right\}$ FOC, $x^2+y^2=10 \}$ C

1) $\frac{2\cancel{\lambda} x}{2\cancel{\lambda}} = \frac{1}{2\lambda}$ $\quad x = 1/2\lambda \quad (\lambda \neq 0)$

2) $\frac{2\lambda y}{2\lambda} = \frac{3}{2\lambda}$ $\quad y = \frac{3}{2\lambda}$

3) $x^2+y^2 = \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 = 10$

$$\frac{1^2+3^2}{(2\lambda)^2} = 10 \quad \Big|\cdot(2\lambda)^2$$

$$\frac{10}{10} = \frac{\cancel{10}\,(2\lambda)^2}{\cancel{10}}$$

$(2\lambda)^2 = 1 \qquad 2\lambda = \pm 1 \qquad \lambda = \pm 1/2$

$\lambda = 1/2$: $x=1, y=3$, $\lambda = 1/2$

$(1,3;1/2)$ $\quad f=10$ max

$\lambda = -1/2$: $x=-1, y=-3$

$(-1,-3;-1/2)$ $\quad f=-10$ min

The first order leading principal minor is $F_{xx} = 6x$ and the second order leading principal minor is $\det D^2F(\mathbf{x}) = -36xy - 81$. At $(0, 0)$, these two minors are 0 and -81, respectively. Since the second order leading principal minor is negative, $(0, 0)$ is a saddle of F — neither a max nor a min. At $(3, -3)$, these two minors are 18 and 243. Since these two numbers are positive, $D^2F(3, -3)$ is positive definite and $(3, -3)$ is a strict local min of F.

Notice that $(3, -3)$ is not a *global* min, because at the point $(0, n)$, $F(0, n) = -n^3$, which goes to $-\infty$ as $n \to \infty$.

EXERCISES

stationary pts.

"

17.1 For each of the following functions defined on $\mathbf{R}^2$, find the critical points and classify these as local max, local min, saddle point, or "can't tell":

a) $x^4 + x^2 - 6xy + 3y^2$, *b)* $x^2 - 6xy + 2y^2 + 10x + 2y - 5$,
c) $xy^2 + x^3y - xy$, *d)* $3x^4 + 3x^2y - y^3$.

17.2 For each of the following functions defined on $\mathbf{R}^3$, find the critical points and classify them as local max, local min, saddle point, or "can't tell":
a) $x^2 + 6xy + y^2 - 3yz + 4z^2 - 10x - 5y - 21z$,
b) $(x^2 + 2y^2 + 3z^2)e^{-(x^2+y^2+z^2)}$.

17.4 GLOBAL MAXIMA AND MINIMA

The first and second order sufficient conditions of the last section will find all the local maxima and minima of a differentiable function whose domain is an open set in $\mathbf{R}^n$. As Example 17.2 illustrates, these conditions say nothing about whether or not any of these local extrema is a *global* max or min. In this section, we will discuss sufficient conditions for global maxima and minima of a real-valued function on $\mathbf{R}^n$.

The study of one-dimensional optimization problems in Section 3.5 put forth two conditions for a critical point x^* of f to be a global max (or min), when f is a C^2 function defined on a connected interval I of $\mathbf{R}^1$:

(1) x^* is a local max (or min) and it's the only critical point of f in I; or
(2) $f'' \leq 0$ on all of I (or $f'' \geq 0$ on I for a min), that is, f is a concave function on I (or f is a convex function for a min).

Condition 1 does not work in higher dimensions, as the function F whose level sets are pictured in Figure 17.1 illustrates. The point A in Figure 17.1 is a local max of F in the open set U. Even though A is the only critical point of F in U, the function F takes on a higher value at point B.

Problems for Lecture 6

BI

1. Find all stationary points and classify them

a) $f(x,y) = e^{xy}$

b) $f(x,y) = e^{x-2y}$

c) $f(x,y) = \sqrt{x^2+y^2+1}$

d) $f(x,y) = x\ln x + y\ln y$

e) ⊛ $f(x,y) = x\ln(y) - y\ln(x)$ (⊛ Difficult.)

2. Solve the Lagrange problems

a) $\max/\min\ f(x,y) = 3x+4y$ when $x^2+y^2=25$

b) $\max\ f(x,y) = y$ when $x^2+y^3=0$

c) $\min\ f(x,y) = 3x^2+4y^2$ when $xy=1$

Solutions for Lecture 6

1. a) $f'_x = ye^{xy}$ $\qquad f''_{xx} = y^2e^{xy}$ $\qquad f''_{xy} = (1+xy)e^{xy}$

$f'_y = xe^{xy}$ $\qquad f''_{yy} = x^2e^{xy}$

$f'_x = f'_y = 0$

$y = x = 0 \Rightarrow$ Stat: $(0,0)$

$H(f)(0,0) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$AC - B^2 = -1 < 0$ saddle pt

b) $u = x - 2y$

$f'_x = e^u \cdot 1$ $\qquad f''_{xx} = e^u \cdot 1$ $\qquad f''_{xy} = e^u \cdot 1 \cdot (-2)$

$f'_y = e^u \cdot (-2)$ $\qquad f''_{yy} = e^u(-2)^2$

$f'_x = f'_y = 0$

$e^{x-2y} = 0$ $\Rightarrow$ no stat. pts

impossible

c) $u = x^2 + y^2 + 1$

$f'_x = \frac{1}{2\sqrt{u}} \cdot 2x = \frac{x}{\sqrt{u}}$

$f'_y = \frac{1}{2\sqrt{u}} \cdot 2y = \frac{y}{\sqrt{u}}$

$f'_x = f'_y = 0$

$x = y = 0 \Rightarrow$ Stat. pts: $(0,0)$

$(u = 1 \neq 0)$

$f''_{xx} = \frac{\left(1 \cdot \sqrt{u} - x \cdot \frac{\cancel{2}x}{\cancel{2}\sqrt{u}}\right)\cancel{2}\sqrt{u}}{u \cdot \cancel{2}\sqrt{u}} = \frac{\cancel{2}u - x^2}{\cancel{2}u\sqrt{u}} = \frac{x^2+y^2+1-x^2}{u\sqrt{u}} = \frac{y^2+1}{u\sqrt{u}}$

$f''_{xy} = \frac{-x \cdot \frac{1}{\cancel{2}\sqrt{u}} \cdot \cancel{2}x}{u} = \frac{-x^2}{u\sqrt{u}}$

$f''_{yy} = \frac{x^2+1}{u\sqrt{u}}$ $\quad \leftarrow$ symmetry $f(y,x) = f(x,y)$

$f''_{yy}(x,y) = f''_{xx}(y,x)$

$H(f)(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow (0,0)$ is local min

$AC - B^2 = 1 > 0, \quad A = 1 > 0$

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c) $f(x,y) = \sqrt{u}$ with $u = x^2+y^2+1$

$$f'_x = \frac{1}{2\sqrt{u}} \cdot 2x = \frac{2x}{2\sqrt{u}} = \underline{\frac{x}{\sqrt{u}}}$$

$$f'_y = \frac{1}{2\sqrt{u}} \cdot 2y = \frac{2y}{2\sqrt{u}} = \underline{\frac{y}{\sqrt{u}}}$$

$f'_x = f'_y = 0$: $\frac{x}{\sqrt{u}} = 0 \Rightarrow x = 0$

$\frac{y}{\sqrt{u}} = 0 \Rightarrow y = 0$

$(u = \sqrt{1} \neq 0)$

$\Rightarrow$ <u>Stat. pts:</u> $(x,y) = \underline{(0,0)}$

$$f''_{xx} = \left(\frac{x}{\sqrt{u}}\right)'_x = \frac{\left(1 \cdot \sqrt{u} - x \cdot \frac{1}{\cancel{2}\sqrt{u}} \cdot \cancel{2}x\right) \cdot \sqrt{u}}{u \cdot \sqrt{u}}$$

$$= \frac{u - x^2}{u\sqrt{u}} = \frac{\cancel{x^2} + y^2 + 1 - \cancel{x^2}}{u\sqrt{u}} = \underline{\frac{y^2+1}{u\sqrt{u}}}$$

$$f''_{xy} = \left(\frac{x}{\sqrt{u}}\right)'_y = \frac{\left(0 \cdot \sqrt{u} - x \cdot \frac{1}{\cancel{2}\sqrt{u}} \cdot \cancel{2}y\right) \cdot \sqrt{u}}{u \cdot \sqrt{u}} = \underline{\frac{-xy}{u\sqrt{u}}}$$

$$f''_{yy} = \left(\frac{y}{\sqrt{u}}\right)'_y = \frac{\left(1 \cdot \sqrt{u} - y \cdot \frac{1}{\cancel{2}\sqrt{u}} \cdot \cancel{2}y\right) \cdot \sqrt{u}}{u \cdot \sqrt{u}} = \frac{u - y^2}{u\sqrt{u}} = \underline{\frac{x^2+1}{u\sqrt{u}}}$$

$$H(f)(0,0) = \begin{pmatrix} 1/1\sqrt{1} & 0/1\sqrt{1} \\ 0/1\sqrt{1} & 1/1\cdot\sqrt{1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$$

$\det H(f)(0,0) = AC - B^2 = 1 > 0$
$A = 1 > 0$
$\Rightarrow$ $(0,0)$ is a <u>local min</u>

d) $f'_x = 1\cdot \ln x + x\cdot \frac{1}{x} = \ln x + 1$

$f'_y = \ln y + 1$

Stat. pts:

$\ln x + 1 = \ln y + 1 = 0$

$x = y = e^{-1} \Rightarrow (x,y) = (e^{-1}, e^{-1})$

$f''_{xx} = 1/x \quad f''_{xy} = 0 \quad f''_{yy} = 1/y$

$H(f)(e^{-1}, e^{-1}) = \begin{pmatrix} e & 0 \\ 0 & e \end{pmatrix} \Rightarrow (e^{-1}, e^{-1})$ is local min

$AC - B^2 = e^2 > 0, \; A = e > 0$

$*$ = Difficult

e) $*$ $f'_x = \ln y - y\cdot\frac{1}{x} = \ln y - \frac{y}{x} = 0$

$f'_y = x\cdot\frac{1}{y} - \ln x = \frac{x}{y} - \ln x = 0$

Stat. pts:

$\ln y = \frac{y}{x} \Rightarrow x = \frac{y}{\ln y} \Rightarrow \ln\left(\frac{y}{\ln y}\right) = \frac{y/\ln y}{y} = \frac{1}{\ln y}$

$\ln x = \frac{x}{y}$

$\ln y \cdot \ln\left(\frac{y}{\ln y}\right) = 1$

$\ln y \cdot (\ln y - \ln(\ln y)) = 1$

$u(y) = \ln(y)\left(\ln y - \ln(\ln y)\right)$

$u' = \frac{1}{y}\left(\ln y - \ln(\ln y)\right) + \ln y \cdot \left(\frac{1}{y} - \frac{1}{\ln y}\cdot\frac{1}{y}\right)$

$= \frac{\ln y - \ln(\ln y) + \ln y - 1}{y}$

$= \frac{2\ln y - \ln(\ln y) - 1}{y}$

To check if $u = 1$ has solutions, find out when u is inc. / dec.

⇓

look at sign of u'

$u' = 0$:

$2\ln y - \ln(\ln y) = 1$

$\ln\left(\frac{y^2}{\ln y}\right) = 1$

$\frac{y^2}{\ln y} = e$

$v = \frac{y^2}{\ln y}$

$v' = \frac{2y\ln y - y^2\cdot\frac{1}{y}}{(\ln y)^2}$

$= \frac{y(2\ln y - 1)}{(\ln y)^2}$

To check if $v = e$ has solutions, find out where v is inc. / dec.

$\Rightarrow$ look at sign of v'.

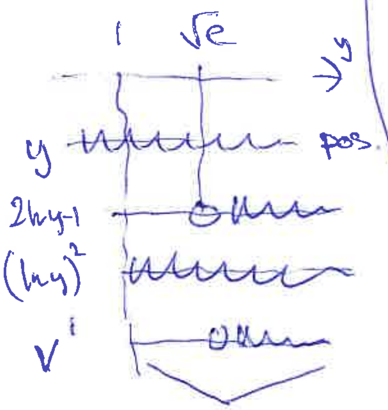
$v' = 0$:

$y\cdot(2\ln y - 1) = 0$

$2\ln y - 1 = 0$

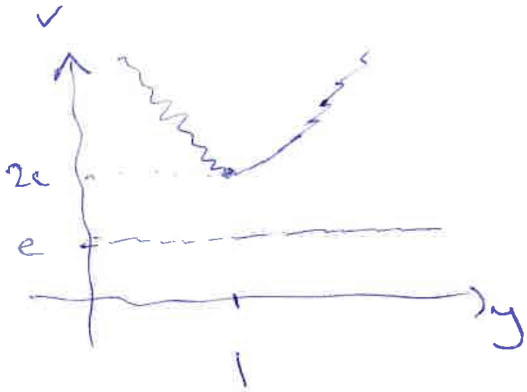
$\ln y = \frac{1}{2}$

$y = e^{1/2} = \sqrt{e}$



min for v:

$y = \sqrt{e} \Rightarrow v = \frac{e}{1/2} = 2e > e$



$$v(y) = \frac{y^2}{\ln y}$$

$v = e$ no solution

$u' = 0$ no solution

$$u' = \frac{2\ln y - \ln(\ln y) - 1}{y}$$

$y > 1$: $y > 0$, $2\ln y - \ln(\ln y) - 1$ const. sign since it is never zer

$y = e \rightsquigarrow 2 - \ln 1 - 1 = 1 > 0$

$\Downarrow$

$u' > 0$ for all $y > 1$

u increasing fn.

$\Downarrow$

$u = 1$ has at most one solution

u inc. function on $1 < y < \infty$

$y = e$ is a solution since

$\ln e\,(\ln(e) - \ln(\ln e)) = 1 \cdot (1 - 0) = 1$

$\Downarrow$

$y = e$ only solution of $u = 1$

$$x = \frac{y}{\ln y} = \frac{e}{\ln e} = e$$

$\Downarrow$

$(x,y) = (e,e)$ unique stat. pt. of f.

$$H(f) = \begin{pmatrix} y/x^2 & 1/y - 1/x \\ 1/y - 1/x & -x/y^2 \end{pmatrix}$$

$$H(f)(e,e) = \begin{pmatrix} 1/e & 0 \\ 0 & 1/e \end{pmatrix}$$

$AC - B^2 = 1/e^2 > 0$

$A = 1/e > 0$

$\Downarrow$

$(x,y) = (e,e)$ is local min

2.

a) $L = 3x + 4y - \lambda\cdot(x^2+y^2)$

FOC $\begin{cases} L'_x = 3 - \lambda\cdot 2x = 0 \\ L'_y = 4 - \lambda\cdot 2y = 0 \end{cases}$ $\Rightarrow x = \dfrac{3}{2\lambda}$, $\Rightarrow y = \dfrac{4}{2\lambda}$

C $\{ \; x^2+y^2 = 25$

$x^2+y^2 = \left(\frac{3}{2\lambda}\right)^2 + \left(\frac{4}{2\lambda}\right)^2 = 25$

$\dfrac{9+16}{4\lambda^2} = 25$

$\dfrac{25}{4\lambda^2} = 25$

$4\lambda^2 = 1$

$\lambda^2 = 1/4$

$\underline{\lambda = \pm 1/2}$

$\underline{\lambda = 1/2}$: $x = 3,\ y = 4$

$\Downarrow$

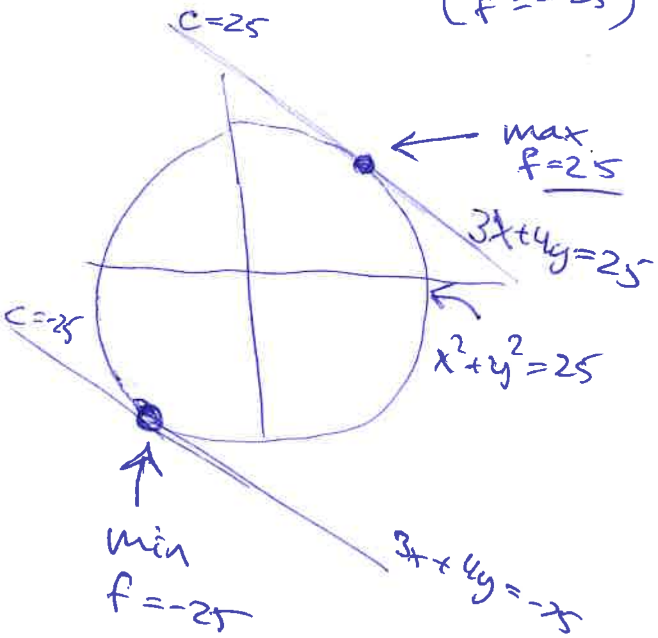
$(x,y;\lambda) = \underline{(3,4;\ 1/2)}$

$(f = 25)$

$\underline{\lambda = -1/2}$: $x = -3,\ y = -4$

$(x,y;\lambda) = \underline{(-3,-4;\ -1/2)}$

$(f = -25)$



inc. values of c means lines = level curves move up and to the right

b) max y when $x^2+y^3=0$

$$\mathcal{L} = y - \lambda\cdot(x^2+y^3)$$

$$\text{FOC}\begin{cases} \mathcal{L}'_x = -\lambda\cdot 2x = 0 \\ \mathcal{L}'_y = 1-\lambda\cdot 3y^2 = 0 \end{cases} \Rightarrow \underline{\lambda = 0} \text{ or } \underline{x=0}$$

$$C \; \{ \; x^2+y^3 = 0$$

$\lambda = 0$: $1=0$ imp.

$x = 0$: $x^2+y^3=0 \Rightarrow y=0$; $1-\lambda\cdot 3y^2=0 \Rightarrow 1=0$ imp.

no solution of FOC+C.

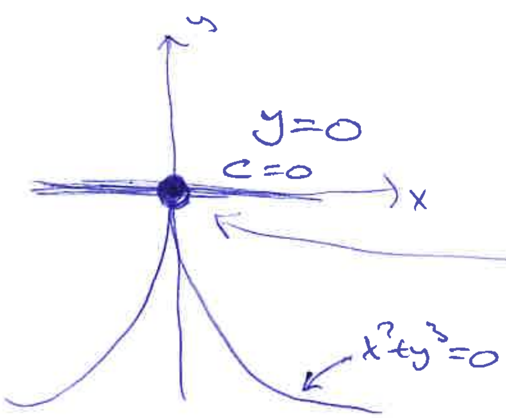
$$\boxed{\begin{aligned} g'_x &= 2x = 0 \\ g'_y &= 3y^2 = 0 \\ x^2&+y^3=0 \end{aligned}}$$

$x=0$

$y=0$

$x=y=0$ ok.

$(0,0)$ is adm. pt. with $g'_x = g'_y = 0$ $\Rightarrow$ can be max



inc. values of c means level curve $y=c$ moves up.

Max = 0

c) min $f = 3x^2+4y^2$ when $xy=1$

$$L = 3x^2+4y^2 - \lambda\cdot xy$$

FOC: $L'_x = \boxed{6x - \lambda y = 0}$

$L'_y = \boxed{8y - \lambda x = 0}$

C: $\boxed{xy = 1}$

① $x = \frac{\lambda y}{6}$

② $8y - \lambda\cdot\left(\frac{\lambda y}{6}\right) = 0 \quad | \cdot 6$

$48y - \lambda^2 y = 0$

$y(48-\lambda^2) = 0$

$y=0$ or $\lambda^2 = 48$

$\lambda = \pm\sqrt{48}$

$y=0$

③ $xy=1$

$x\cdot 0 = 1$

imp.

no soln.

$\lambda = \sqrt{48}$

① $x = \frac{\sqrt{48}}{6}y$

③ $xy = \frac{\sqrt{48}}{6}y\cdot y = 1$

$y^2 = \frac{6}{\sqrt{48}} = \frac{\cancel{2}\cdot 3}{\sqrt{4}\cdot\sqrt{12}}$

$= \frac{3}{\sqrt{12}} = \frac{\sqrt{3}\cdot\sqrt{3}}{\sqrt{3}\cdot\sqrt{4}}$

$= \sqrt{3/4}$

$y = \pm\sqrt[4]{3/4}$

$x = \pm\sqrt[4]{3/4}\cdot\frac{\sqrt{48}}{6}$

$= \pm\sqrt[4]{4/3}$

$\pm\sqrt[4]{3/4}\cdot\frac{\sqrt{48}}{6}$

$= \pm\sqrt[4]{\frac{3}{4}}\cdot\sqrt[4]{\left(\frac{4}{3}\right)^2}$

$= \pm\sqrt[4]{4/3}$

$\lambda = -\sqrt{48}$

① $x = -\frac{\sqrt{48}}{6}y$

③ $xy = -\frac{\sqrt{48}}{6}y\cdot y = 1$

$y^2 = -\frac{6}{\sqrt{48}}$

imp.

no soln.

Pts:

$\left(\sqrt[4]{4/3}, \sqrt[4]{3/4}; \sqrt{48}\right)$

$\left(-\sqrt[4]{4/3}, -\sqrt[4]{3/4}; \sqrt{48}\right)$

$f = 3\cdot\sqrt{4/3} + 4\cdot\sqrt{3/4}$

≈ 6.93 min pt./value

inc. values of c means the level curve = ellipse $3x^2+4y^2=c$ moves outwards ("radius" increases)