Plan

1. Introduction
2. Solving linear systems of equation
3. Key method: Gaussian elimination

2 Solving linear systems

Ex:

$$\begin{aligned} 2x - 5y &= 12 \\ x + 3y &= -14 \end{aligned}$$

2×2 linear system

$\#$ eqns $= \#$ vars

$$\begin{aligned} x + y + z + w &= 4 \\ x - y + 2z \quad &= 7 \\ 2x \quad + 3z + w &= 10 \end{aligned}$$

3×4 linear system

Solution techniques:

- substitution methods
- elimination methods

$$\begin{cases} 2x - 5y = 12 & (1) \\ x + 3y = -14 & (2) \end{cases}$$

$(2): \; x = -14 - 3y$

$\cancel{(1): 2(-14-3y) + 3y = -14}$

$\cancel{-28 - 6y + 3y = -14}$

$$\begin{aligned} (1) \quad 2(-14-3y) - 5y &= 12 \\ -28 - 6y - 5y &= 12 \\ -11y &= 40 \\ y &= \frac{40}{-11} \end{aligned}$$

$$x = -14 - 3 \cdot \left(-\frac{40}{11}\right) = \frac{-14 \cdot 11 + 120}{11} = \frac{-34}{11}$$

$$(x, y) = \left(-\frac{34}{11}, -\frac{40}{11}\right)$$

Ex: elimination

$$2x - 3y = 2$$
$$x + 4y = 16 \quad | \cdot (-2)$$
$$\downarrow$$

(1) $2x - 3y = 2$
(2) $-2x - 8y = -32$

(1) $2x - 3y = 2$
(1)+(2) $\underline{-11y = -30}$

$2x - 3 \cdot 30/11 = 2$
$y = 30/11$

$$2x = \frac{22}{11} + \frac{90}{11} = \frac{112}{11}$$
$$x = \frac{56}{11}$$
$$(x, y) = \underline{\underline{(56/11, 30/11)}}$$

Geometry of linear systems

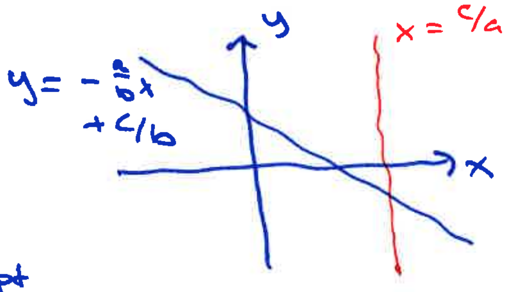
Linear eqn: $a_1 x_1 + a_2 \cdot x_2 + \dots + a_n x_n = b$

($a_1, a_2, \dots, a_n, b$ are given numbers, $x_1, x_2, \dots, x_n$ variables)

$\underline{n=2}$: $ax + by = c$

$b \neq 0$: $by = -ax + c$

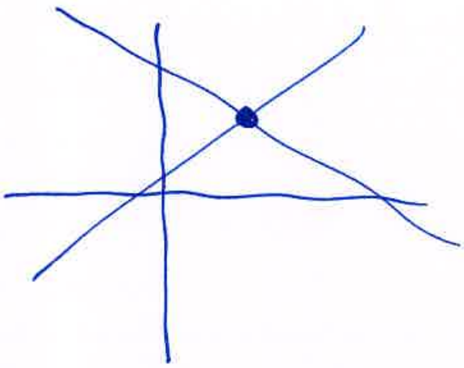
$$y = \underbrace{-\frac{a}{b}}_{\text{slope}} x + \underbrace{\frac{c}{b}}_{\text{y-intercept}}$$



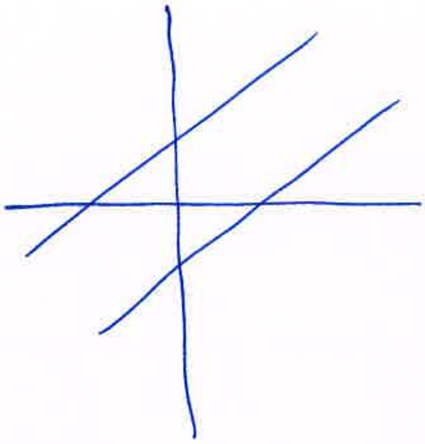
$\underline{b=0}$: $ax = c$
$a \neq 0 \Downarrow$
$x = c/a$

2x2 linear system:

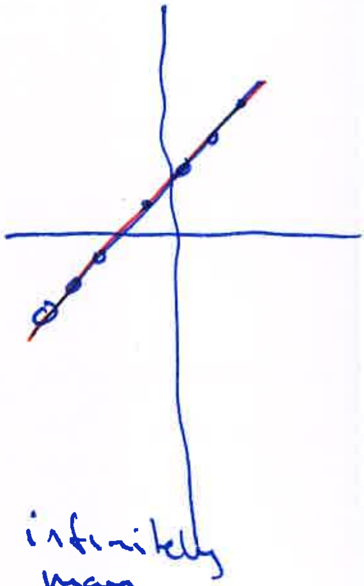
$$ax+by=c$$
$$dx+ey=f$$



one solution



no solutions



infinitely many solutions

In general:

i) The graph of a linear equation does not curve

ii) The solution of a linear system is either

i) one solution

ii) infinitely many solutions

iii) no solutions

is not possible for linear systems

③ Gaussian elimination

- general method for solving any linear system

Ex:
$$x+y+z=3$$
$$x+2y+4z=7$$
$$x+3y+9z=13$$

3x3 linear system

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right) \begin{array}{l} \cdot(-1) \\ + \\ \\ \end{array}$$

augmented matrix of the system

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{pmatrix}$$ is called the coeff. matrix of the system

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 1 & 3 & 9 & 13 \end{array}\right) \begin{array}{l} \cdot(-1) \\ \\ + \end{array}$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right) \begin{array}{l} \\ -2 \\ + \end{array}$$

$$\downarrow$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

echelon form

Elementary row operations:

i) switch two rows
ii) multiply a row with $c \neq 0$
iii) add a multiple of one row to another row

Echelon form:

* a pivot: the first non-zero number in a row
* a matrix is in echelon form if
  i) all zero rows are in the bottom of the matrix
  ii) all entries under a pivot are zero

$$x + y + z = 3$$
$$y + 3z = 4$$
$$2z = 2 \rightarrow z = 1$$

Back substitution:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

(3) $2z = 2$
$\underline{z = 1}$

(2) $y + 3z = 4$
$y + 3 \cdot 1 = 4$
$\underline{y = 1}$

(1) $x + y + z = 3$
$x + 1 + 1 = 3$
$\underline{x = 1}$

One solution: $(x, y, z) = \underline{\underline{(1, 1, 1)}}$

Summary:

linear system → augmented matrix

augmented matrix → (elementary row operations) → echelon form

echelon form → linear system

linear system → (back substitution) → Solutions

Facts: i) any matrix can be transformed into an echelon form using elementary row operations

ii) the echelon form is not unique, but the pivot positions are.

Reduced echelon form:

A matrix is in reduced echelon form if

i) all zero rows are at the bottom
ii) all entries under a pivot are zero
iii) all pivots are 1
iv) all entries over a pivot are zero

Ex:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right) \to \cdots \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right) \cdot \tfrac{1}{2}$$

0

echelon form

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right) \begin{array}{c} \\ + \\ -3 \end{array} \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \begin{array}{c} + \\ \\ -1 \end{array}$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right) \begin{array}{c} + \\ -1 \\ \\ \end{array} \to \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right)$$

reduced echelon form

$$x = 1$$
$$y = 1$$
$$z = 1$$

Fact:

(3) The reduced echelon form is unique

Gauss-Jordan elimination

Ex:

$$\begin{aligned} x+y+4z-w&=7\\ 2x-y+z+4w&=3\\ x+4y+11z-7w&=18 \end{aligned}$$

$$\left(\begin{array}{cccc|c} 1 & 1 & 4 & -1 & 7\\ 2 & -1 & 1 & 4 & 3\\ 1 & 4 & 11 & -7 & 18 \end{array}\right) \begin{array}{l} -2 \quad -1\\ +\\ \quad\ + \end{array}$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 4 & -1 & 7\\ 0 & -3 & -7 & 6 & -11\\ 0 & 3 & 7 & -6 & 11 \end{array}\right) \begin{array}{l} 1\\ +\end{array}$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & 4 & -1 & 7\\ 0 & -3 & -7 & 6 & -11\\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\begin{array}{cccc} b & b & f & f \end{array}$$

echelon form

$$\begin{aligned} x+y+4z-w&=7\\ -3y-7z+6w&=-11 \end{aligned}$$

x, y: basic

z, w: free

$$\frac{-3y}{-3}=\frac{7z-6w-11}{-3} \qquad y=\frac{11}{3}-\frac{7}{3}z+2w$$

$$\begin{aligned} x+y+4z-w&=7\\ x&=7-y-4z+w\\ &=7-\left(\frac{11}{3}-\frac{7}{3}z+2w\right)-4z+w\\ &=\frac{10}{3}-\frac{5}{3}z-w \end{aligned}$$

Solutions: $(x,y,z,w) = \left(\frac{10}{3} - \frac{5}{3}z - w,\ \frac{11}{3} - \frac{7}{3}z + 2w,\ z, w \right)$

where z, w are free

infinitely many solutions

Ex:

$$\begin{aligned} x + 2y - z &= 7 \\ 2x - y + 3z &= 11 \\ x + 7y - 6z &= 9 \end{aligned}$$

$$\begin{array}{c} \left(\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 2 & -1 & 3 & 11 \\ 1 & 7 & -6 & 9 \end{array} \right) \end{array} \begin{array}{l} -2 \\ -1 \\ \end{array}$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 0 & -5 & 5 & -3 \\ 0 & 5 & -5 & 2 \end{array} \right) \begin{array}{l} \\ 1 \\ \end{array}$$

$$\to \left(\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 0 & -5 & 5 & -3 \\ 0 & 0 & 0 & -1 \end{array} \right)$$

echelon form

no solutions

$$\begin{aligned} x + 2y - z &= 7 \\ -5y + 5z &= -3 \\ 0 &= -1 \end{aligned}$$

Summary: Solutions of linear systems

i) no solutions $\Longleftrightarrow$ there is a pivot in the last column

ii) Otherwise:

columns with pivot $\leftrightarrow$ basic variables

— " — without pivot $\leftrightarrow$ free variables

No free variables: one solution

At least one free variable: infinitely many solutions

Ex:

$$\begin{aligned} 2y + 2z &= 2 \\ 2x \quad + 2z &= 4 \\ 2x + 2y \quad &= 6 \end{aligned}$$

$$\left(\begin{array}{ccc|c} 0 & 2 & 2 & 2 \\ 2 & 0 & 2 & 4 \\ 2 & 2 & 0 & 6 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & 0 & 2 & 4 \\ 0 & 2 & 2 & 2 \\ 2 & 2 & 0 & 6 \end{array}\right) -1$$

$$\to \left(\begin{array}{ccc|c} 2 & 0 & 2 & 4 \\ 0 & 2 & 2 & 2 \\ 0 & 2 & -2 & 2 \end{array}\right) -1 \to \left(\begin{array}{ccc|c} 2 & 0 & 2 & 4 \\ 0 & 2 & 2 & 2 \\ 0 & 0 & -4 & 0 \end{array}\right)$$

$$\left.\begin{aligned} 2x + 2z &= 4 \\ 2y + 2z &= 2 \\ -4z &= 0 \end{aligned} \quad \begin{aligned} x &= 2 \\ y &= 1 \\ z &= 0 \end{aligned}\right\} \Rightarrow (x, y, z) = (2, 1, 0)$$