Plan

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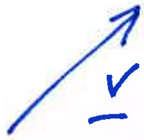
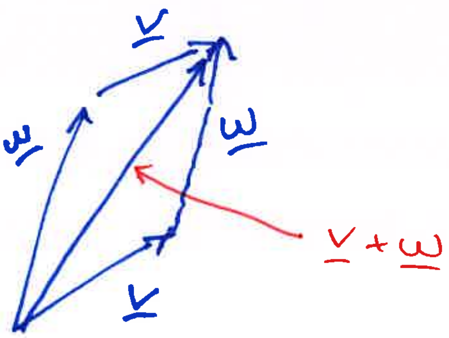
1 Vectors and matrices

Vector: quantity with magnitude and direction

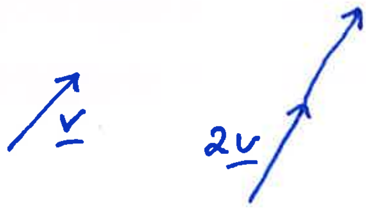


$\|\underline{v}\|$ = length of the vector $\underline{v}$

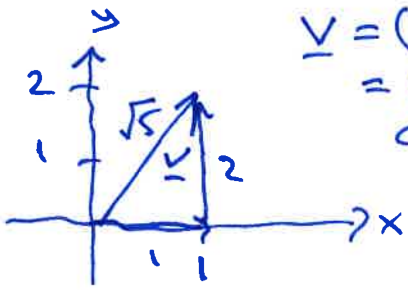
Addition:



Scalar multiplication



Coordinates:



$\underline{v} = (1,2)$
$= (v_x, v_y)$ components

Some vector operations using coordinates:

$\underline{v} = (v_x, v_y)$

i) $\|\underline{v}\| = \sqrt{v_x^2 + v_y^2}$

ii) $\underline{v} + \underline{w} = (v_x, v_y) + (w_x, w_y)$
$= (v_x + w_x, v_y + w_y)$

iii) $r \cdot \underline{v} = r \cdot (v_x, v_y)$
$= (rv_x, rv_y)$

Defn: An <u>n-vector</u> is an n-tuple

$$\underline{v} = (v_1, v_2, \ldots, v_n)$$

Operations:

i) <u>Addition:</u> $(v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n)$

$$= (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$$

ii) <u>Scalar multiplication</u> $r \cdot (v_1, v_2, \ldots, v_n)$

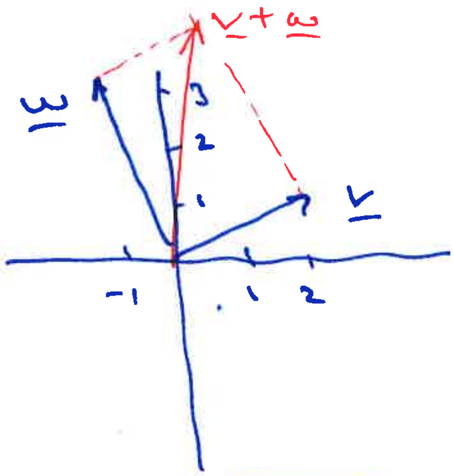
$$= (rv_1, rv_2, \ldots, rv_n)$$

iii) <u>Length:</u> $\|(v_1, v_2, \ldots, v_n)\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$

Ex: $\underline{v} = (2,1)$

$\underline{w} = (-1,3)$

$\underline{v} + \underline{w} = (1,4)$



$r \cdot \underline{v}$

$\|\underline{v}\| = \sqrt{2^2 + 1^2} = \sqrt{5}$

$\|\underline{w}\| = \sqrt{(-1)^2 + 3^2} = \sqrt{10}$

$\|\underline{v} + \underline{w}\| = \sqrt{1^2 + 4^2} = \sqrt{17}$

Column vectors:

$\underline{v} = (2,1)$

$\updownarrow$

$\underline{v} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\underline{v} = (1,2,4,-1)$

$\updownarrow$

$\underline{v} = \begin{pmatrix} 1 \\ 2 \\ 4 \\ -1 \end{pmatrix}$

Matrices: An $m \times n$ - matrix A is a rectangular array of numbers (m rows, n columns)

Ex: $A = \begin{pmatrix} 1 & 7 & 3 \\ -1 & 0 & 5 \end{pmatrix}$ 2x3-matrix

$B = \begin{pmatrix} 7 & 2 \\ 3 & 1 \end{pmatrix}$ 2x2-matrix (square)

Operation on matrices:

i) Addition: (subtraction) $A+B$, $A-B$ (defined if A and B have the same size)

Ex: $\begin{pmatrix} 1 & 7 & 3 \\ -1 & 0 & 5 \end{pmatrix} + \begin{pmatrix} 1 & 1 & 1 \\ 1 & 7 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 8 & 4 \\ 0 & 7 & 6 \end{pmatrix}$

$\begin{pmatrix} 2 \\ 1 \end{pmatrix} + \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \end{pmatrix}$

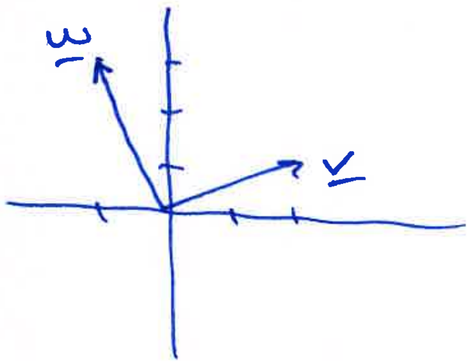
$\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

ii) Scalar multiplication: $r \cdot A$

Ex: $3 \cdot \begin{pmatrix} 1 & 7 & 3 \\ -1 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 3 & 21 & 9 \\ -3 & 0 & 15 \end{pmatrix}$

Inner product of vectors (dot product, scalar product)

Ex: $\underline{v} = (2,1)$
$\underline{w} = (-1,3)$



Inner product:

$$\underline{v} \cdot \underline{w} = (2,1) \cdot (-1,3) = 2 \cdot (-1) + 1 \cdot 3 = \underline{\underline{1}}$$

Defn: We write $\underline{v} \perp \underline{w}$ when the vectors are perpendicular (angle of $90°$)

Fact: $\underline{v} \perp \underline{w} \iff \underline{v} \cdot \underline{w} = 0$

$\underline{v} \cdot \underline{w} > 0$: angle $< 90°$

$\underline{v} \cdot \underline{w} < 0$: angle $> 90°$

In general:

$$\underline{v} \cdot \underline{w} = (v_1, v_2, \ldots, v_n) \cdot (w_1, w_2, \ldots, w_n) = v_1 \cdot w_1 + v_2 \cdot w_2 + \ldots + v_n \cdot w_n$$

2 Matrix multiplication

A, B matrices $\qquad A \cdot B$ defined if $\#\,\text{cols}(A) = \#\,\text{rows}(B)$

$\underset{m\times n}{A} \quad \underset{n\times p}{B}$

result is matrix $(m \times p)$

components are inner products of rows in A and cols in B

Ex:
$$\begin{pmatrix} 2 & 3 \\ 5 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \cdot \\ \cdot \end{pmatrix} = \begin{pmatrix} 8 \\ 3 \end{pmatrix}$$

2×2 — 2×1

$(2,3)\cdot(1,2) = 2\cdot 1 + 3 \cdot 2 = 8$

$(5,-1)\cdot(1,2) = 5 \cdot 1 + (-1)\cdot 2 = 3$

Ex:
$$\begin{pmatrix} 2 & 4 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 3 & 7 & 0 \\ 2 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 14 & 18 & 8 \\ 3 & 7 & 0 \end{pmatrix}$$

2×2 —— 2×3

Notice: $\boxed{A \cdot B \neq B \cdot A}$

non-commutative matrix multiplication

$$\begin{pmatrix} 3 & 7 & 0 \\ 2 & 1 & 2 \end{pmatrix} \cdot \begin{pmatrix} 2 & 4 \\ 1 & 0 \end{pmatrix}$$

$2 \times 3 \neq 2 \times 2$

not defined

Ex: $(A+B)^2 = (A+B)\cdot(A+B)$

$= A\cdot A + A \cdot B + B \cdot A + B \cdot B$

$= A^2 + AB + BA + B^2$

Matrix multiplication and linear systems

Ex:
$$\left.\begin{aligned} x+2y-z &= 7 \\ 4x-y+2z &= 0 \\ 2x+y+z &= 4 \end{aligned}\right\}$$

Gauss:
$$\left(\begin{array}{ccc|c} 1 & 2 & -1 & 7 \\ 4 & -1 & 2 & 0 \\ 2 & 1 & 1 & 4 \end{array}\right)$$

Coefficient matrix of the system

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 4 & -1 & 2 \\ 2 & 1 & 1 \end{pmatrix} \qquad \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 7 \\ 0 \\ 4 \end{pmatrix}$$

$$A \cdot \underline{x} = \underset{3\times 3}{\begin{pmatrix} 1 & 2 & -1 \\ 4 & -1 & 2 \\ 2 & 1 & 1 \end{pmatrix}} \cdot \underset{3\times 1}{\begin{pmatrix} x \\ y \\ z \end{pmatrix}} = \begin{pmatrix} x+2y-z \\ 4x-y+2z \\ 2x+y+z \end{pmatrix} = \begin{pmatrix} 7 \\ 0 \\ 4 \end{pmatrix} = \underline{b}$$

Matrix form of the linear system: $\boxed{A \cdot \underline{x} = \underline{b}}$

Powers of matrices: defined if A square

$A^2 = A \cdot A$

$A^3 = A \cdot A \cdot A = A^2 \cdot A$

$A^4 = A \cdot A \cdot A \cdot A = A^2 \cdot A^2 = A^3 \cdot A$

Identity matrix: $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\quad I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ ...

Fact: $A \cdot I = A$, $I \cdot A = A$ for any matrix A

$A = \begin{pmatrix} 2 & 3 \\ 7 & -1 \end{pmatrix}$:

$A \cdot I = \begin{pmatrix} 2 & 3 \\ 7 & -1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 7 & -1 \end{pmatrix} = A$

$I \cdot A = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 3 \\ 7 & -1 \end{pmatrix} = \begin{pmatrix} 2 & 3 \\ 7 & -1 \end{pmatrix} = A$

Ex: $\boxed{A \cdot \underline{x} = \underline{b}}$ lin system in matrix form

Solve = find $\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$

$A^{-1} \cdot A\underline{x} = A^{-1} \cdot \underline{b}$

$I \cdot \underline{x} = A^{-1} \cdot \underline{b}$

$\underline{x} = A^{-1} \cdot \underline{b}$

$2x = 18$

$\frac{1}{2} \cdot 2x = 18 \cdot \frac{1}{2}$

$\underline{x = 9}$

3 Inverse matrices

Let A be a matrix.

Defn: An inverse A^{-1} of A is a matrix such that

$$A^{-1} \cdot A = I$$

and

$$A \cdot A^{-1} = I$$

Ex: $A = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$:

$$\overset{A^{-1}}{\begin{pmatrix} a & b \\ c & d \end{pmatrix}} \cdot \overset{A}{\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$2a + b = 1$ $\quad$ $4a + 3b = 0$ $\quad$ $2a + 4c = 1$ $\quad$ $2b + 4d = 0$

$2c + d = 0$ $\quad$ $4c + 3d = 1$ $\quad$ $a + 3c = 0$ $\quad$ $b + 3d = 1$

<u>Facts:</u> – if A is not <u>square</u>, it <u>cannot</u> have an inverse

– if A has an inverse, then it is unique

<u>Method:</u> A square matrix < is there an inverse A^{-1}? / if yes, then what is A^{-1}

i) Form the matrix $(A \mid I)$

ii) Reduce it to <u>reduced echelon form</u>

$(I \mid \underline{A^{-1}})$ — A^{-1} does exist

$(\neq I \mid *)$ — A^{-1} does not exist

<u>Ex:</u> $A = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$

$$\left(\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{array}\right) \quad -1, \; -\tfrac{1}{2}, \; + \quad \to \ldots$$

$$\downarrow$$

$$\left(\begin{array}{cc|cc} 1 & 1 & 1 & -1 \\ 1 & 3 & 0 & 1 \end{array}\right) -1 \to \left(\begin{array}{cc|cc} 1 & 1 & 1 & -1 \\ 0 & 2 & -1 & 2 \end{array}\right) \cdot \tfrac{1}{2}$$

$$\to \left(\begin{array}{cc|cc} 1 & 1 & 1 & -1 \\ 0 & 1 & -\tfrac{1}{2} & 1 \end{array}\right) -1 \to \left(\begin{array}{cc|cc} 1 & 0 & 3/2 & -2 \\ 0 & 1 & -1/2 & 1 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} 3/2 & -2 \\ -1/2 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 3/2 & -2 \\ -1/2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 3/2 & -2 \\ -1/2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Special case: $n = 2$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $\begin{cases} A^{-1} = \dfrac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} &, ad-bc \neq 0 \\ A^{-1} \text{ does not exist} &, ad-bc = 0 \end{cases}$

$A = \begin{pmatrix} 2 & 4 \\ 1 & 3 \end{pmatrix}$: $ad - bc = 2 \cdot 3 - 4 \cdot 1 = 2$

$$A^{-1} = \frac{1}{2} \cdot \begin{pmatrix} 3 & -4 \\ -1 & 2 \end{pmatrix}$$

$$= \underline{\underline{\begin{pmatrix} 3/2 & -2 \\ -1/2 & 1 \end{pmatrix}}}$$