Chapter 1
Solutions to textbook problems

1.1 Linear systems and Gaussian elimination

Solution 1.1.1. The coefficient matrix A and the augmented matrix $(A|\mathbf{b})$ are:

a) $A = \begin{pmatrix} 7 & 2 \\ 4 & -4 \end{pmatrix}, \quad (A|\mathbf{b}) = \left(\begin{array}{cc|c} 7 & 2 & 4 \\ 4 & -4 & 7 \end{array}\right)$

b) $A = \begin{pmatrix} 0 & 1 & -1 & 1 \\ 1 & 1 & 1 & -2 \\ 1 & -4 & 7 & 3 \end{pmatrix}, \quad (A|\mathbf{b}) = \left(\begin{array}{cccc|c} 0 & 1 & -1 & 1 & 1 \\ 1 & 1 & 1 & -2 & 3 \\ 1 & -4 & 7 & 3 & 14 \end{array}\right)$

Solution 1.1.2. The linear systems are:

a) $\begin{aligned} 2x + y &= 7 \\ x + 3y + 4z &= 5 \\ 5x - 4y + 2z &= 13 \end{aligned}$

b) $\begin{aligned} x + y + z &= 7 \\ x + 2y + 4z &= 12 \\ x + 3y + 9z &= 19 \end{aligned}$

c) $\begin{aligned} x + y - z + w &= 12 \\ 2x - 3y + 4z + 7w &= 10 \end{aligned}$

Solution 1.1.3. We first use substitution to solve the system. The first equation gives $x = 7 - y - z$. We substitute this expression in the last two equations, and this gives the equations

$$y + 3z = 5, \quad 2y + 8z = 12$$

Next we solve the first of these new equations for y, and get $y = 5 - 3z$. When we substitute this in the second equation, we get $2(5 - 3z) + 8z = 12$, or $2z = 2$. This means that $z = 1$, that $y = 5 - 3(1) = 2$, and that $x = 7 - 2 - 1 = 4$. The system has one solution $(x, y, z) = (4, 2, 1)$.
Then we solve the system by Gaussian elimination. We write down the augmented matrix and use the Gaussian process

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 1 & 2 & 4 & 12 \\ 1 & 3 & 9 & 19 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 3 & 5 \\ 0 & 2 & 8 & 12 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 7 \\ 0 & 1 & 3 & 5 \\ 0 & 0 & 2 & 2 \end{array}\right)$$

This is an echelon form, and back substitution gives $2z = 2$, or $z = 1$ in the last equation, $y+3z=5$, or $y=5-3(1)=2$ in the second equation and $x+y+z=7$, or $x=7-2-1=4$ in the first equation. Hence we find one solution $(x,y,z)=(4,2,1)$ also with Gaussian elimination.

Solution 1.1.4. We solve the system by Gaussian elimination, and mark the pivots:

a) We use the Gaussian process

$$\left(\begin{array}{rrr|r} -4 & 6 & 4 & 4 \\ 2 & -1 & 1 & 1 \end{array}\right) \to \left(\begin{array}{rrr|r} -4 & 6 & 4 & 4 \\ 0 & 2 & 3 & 3 \end{array}\right)$$

We find an echelon form, and we see from the pivot positions that there is one free variable, and infinitely many solutions. The free variable is z, and back substitution gives $2y = 3-3z$, or $y = 3/2-3z/2$ in the second equation, and $-4x = 4-6y-4z = 4-6(3/2-3z/2)-4z = -5+5z$, or $x = 5/4-5z/4$ in the first equation. The solutions are $(x,y,z) = (5/4-5z/4, 3/2-3z/2, z)$ with z free.

b) We use the Gaussian process

$$\left(\begin{array}{rr|r} 6 & 1 & 7 \\ 3 & 1 & 4 \\ -6 & -2 & 1 \end{array}\right) \to \left(\begin{array}{rr|r} 6 & 1 & 7 \\ 0 & 1/2 & 1/2 \\ 0 & -1 & 8 \end{array}\right) \to \left(\begin{array}{rr|r} 6 & 1 & 7 \\ 0 & 1 & 1 \\ 0 & 0 & 9 \end{array}\right)$$

We find an echelon form, and since there is a pivot in the last column, there are no solutions.

Solution 1.1.5. We solve the linear systems by Gaussian elimination, and give a geometric description of the solutions:

a) The linear system $3x - 4y = 6$ is already in echelon form, with augmented matrix $\left(\begin{array}{rr|r} 3 & 4 & 6 \end{array}\right)$. It has infinitely many solutions, since x is a basic variable and y is free, and we have that $3x = 4y+6$, which means that $x = 4y/3+2$. The set of solutions is $V = \{(4y/3+2, y) : y \text{ free }\}$. This is a line in $\mathbb{R}^2$.

b) We use a Gaussian process to obtain an echelon form of the system, and get

$$\left(\begin{array}{rrr|r} 1 & 1 & 1 & 4 \\ 2 & -1 & 3 & 3 \\ 3 & 0 & 4 & 7 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 1 & 1 & 4 \\ 0 & -3 & 1 & -5 \\ 0 & -3 & 1 & -5 \end{array}\right) \to \left(\begin{array}{rrr|r} 1 & 1 & 1 & 4 \\ 0 & -3 & 1 & -5 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

The system has infinitely many solutions, since x and y are basic variables and z is free. We have that $-3y+z=-5$, which gives $y = 5/3+z/3$, and $x+y+z=4$, which gives $x = 4-(5/3+z/3)-z = 7/3-4z/3$. The set of solutions is $V = \{(7/3-4z/3, 5/3+z/3, z) : z \text{ free }\}$. This is a line in $\mathbb{R}^3$.

c) We use a Gaussian process to obtain an echelon form of the system, and get

$$\left(\begin{array}{ccc|c} 2 & 1 & -4 & 3 \\ 3 & -2 & 1 & 1 \end{array}\right) \to \left(\begin{array}{ccc|c} 6 & 3 & -12 & 9 \\ 6 & -4 & 2 & 2 \end{array}\right) \to \left(\begin{array}{ccc|c} 6 & 3 & -12 & 9 \\ 0 & -7 & 14 & -7 \end{array}\right)$$

The system has infinitely many solutions, since x and y are basic variables and z is free. We have that $-7y+14z=-7$, which gives $y=1+2z$, and $6x+3y-12z=9$, which gives $6x=9-3(1+2z)+12z=6+6z$, or $x=1+z$. The set of solutions is $V=\{(1+z,1+2z,z) : z \text{ free}\}$. This is a line in $\mathbb{R}^3$.

Solution 1.1.6. We solve the linear systems by Gaussian elimination and give a geometric description of the solutions for each value of h:

a) The linear system $hx - hy = 3$ is already in echelon form, with augmented matrix $\left(\begin{array}{cc|c} h & -h & 3 \end{array}\right)$. If $h=0$, then there is a pivot in the last column, and there are no solutions. If $h \neq 0$, then there is a pivot in the x-column, and there are infinitely many solutions since x is a basic variable and y is free. In this case, we have that $hx = hy+3$, which means that $x = y+3/h$. The set of solutions is $V=\{(y+3/h,y) : y \text{ free}\}$. This is a line in $\mathbb{R}^2$ for each $h \neq 0$.

b) We use a Gaussian process to obtain an echelon form of the system, and get

$$\left(\begin{array}{cc|c} 1 & 2 & h \\ 3 & 6 & 12 \end{array}\right) \to \left(\begin{array}{cc|c} 1 & 2 & h \\ 0 & 0 & 12-3h \end{array}\right)$$

There is a pivot in the x-column, and if $12-3h \neq 0$, then there is a pivot in the last column. This means that if $h \neq 4$, there are no solutions. If $h=4$, then there are infinitely many solutions, since x is a basic variable and y is free. We have that $x+2y=h$ with $h=4$, which gives $x=4-2y$. The set of solutions is $V=\{(4-2y,y) : y \text{ free}\}$. This is a line in $\mathbb{R}^2$ for $h=4$.

c) We use a Gaussian process to obtain an echelon form of the system, and get

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 2 & -1 & 3 & 4 \\ 1 & 7 & -1 & h \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & 2 \\ 0 & 6 & -2 & h-1 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & -3 & 1 & 2 \\ 0 & 0 & 0 & h+3 \end{array}\right)$$

There are pivots in the x- and y-columns, and a pivot in the last column if $h+3 \neq 0$. This means that if $h \neq -3$, there are no solutions. If $h=-3$, then there are infinitely many solutions, since x and y are basic variables and z is free. We have that $-3y+z=2$, which gives $-3y=2-z$, or $y=z/3-2/3$, and $x+y+z=1$, which gives $x=1-(z/3-2/3)-z=5/3-4z/3$. The set of solutions is $V=\{(5/3-4z/3,z/3-2/3,z) : z \text{ free}\}$. This is a line in $\mathbb{R}^3$ for $h=-3$.

Solution 1.1.7. We first use a Gaussian process that gives an echelon form of the matrix:

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & 2 & 4 & 7 \\ 1 & 3 & 9 & 13 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 2 & 8 & 10 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & 1 & 3 & 4 \\ 0 & 0 & 2 & 2 \end{array}\right)$$