f) $ABC = AB \cdot C = \begin{pmatrix} 2 & 2 & 2 \\ 6 & 5 & 3 \\ 0 & 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 8 \\ 6 & 1 & 17 \\ 0 & -2 & 2 \end{pmatrix}$

In e) we could also have used that $(A+B)^2 = (A+B)(A+B) = A^2 + AB + BA + B^2$ and the products computed in a) - d).

Solution 1.3.2. The matrix A is symmetric and the matrix B is not symmetric:

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 7 & 1 \\ 3 & 7 & 1 & 4 \\ 4 & 1 & 4 & 3 \end{pmatrix}, \quad B = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 1 & 7 & 5 \\ 2 & 6 & 2 & 4 \\ 4 & 1 & 4 & 3 \end{pmatrix} \neq B^T$$

Solution 1.3.3. We have

a) $AB(BC - CB) + (CA - AB)BC + CA(A - B)C = AB^2C - ABCB + CABC - AB^2C + CA^2C - CABC = -ABCB + CA^2C$

b) $(A-B)(C-A) + (C-B)(A-C) + (C-A)^2 = AC - BC - A^2 + BA + CA - BA - C^2 + BC + C^2 - CA - AC + A^2 = 0$

Solution 1.3.4. We find:

a) The linear system is

$$\begin{aligned} 3x + y + 5z &= 4 \\ 5x - 3y + 2z &= -2 \\ 4x - 3y - z &= -1 \end{aligned}$$

b) We have that A is invertible since $|A| \neq 0$:

$$|A| = \begin{vmatrix} 3 & 1 & 5 \\ 5 & -3 & 2 \\ 4 & -3 & -1 \end{vmatrix} = 3(3+6) - 1(-5-8) + 5(-15+12) = 27 + 13 - 15 = 25$$

The inverse matrix is therefore given by

$$A^{-1} = \frac{1}{|A|} \begin{pmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{pmatrix}^T = \frac{1}{25} \begin{pmatrix} 9 & 13 & -3 \\ -14 & -23 & 13 \\ 17 & 19 & -14 \end{pmatrix}^T = \frac{1}{25} \begin{pmatrix} 9 & -14 & 17 \\ 13 & -23 & 19 \\ -3 & 13 & -14 \end{pmatrix}$$

c) Since A is invertible, the linear system has one unique solution.

Solution 1.3.5. We have

$$A^T A = \begin{pmatrix} 1 & 2 \\ 2 & -1 \\ 3 & 6 \\ 4 & 5 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 15 & 14 \\ 0 & 5 & 0 & 3 \\ 15 & 0 & 45 & 42 \\ 14 & 3 & 42 & 41 \end{pmatrix}$$

Solution 1.3.6. Using cofactor expansion along the first column and third column to compute $|A|$, we get:

$$\begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 0 & 8 \end{vmatrix} = 1(40) - 0 + 1(12 - 15) = 37, \quad \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 0 & 8 \end{vmatrix} = 1(-3) - 0 + 8(5) = 37$$

We see that the two computations are different, but give the same result. They have roughly the same complexity in this case. If we had computed the determinant using cofactor expansion along another row or column, the computations might have been longer.

Solution 1.3.7. We use elementary row operations that do not change the determinant:

a)

$$\begin{vmatrix} 3 & 1 & 5 \\ 9 & 3 & 15 \\ -3 & -1 & -5 \end{vmatrix} = \begin{vmatrix} 3 & 1 & 5 \\ 0 & 0 & 0 \\ -3 & -1 & -5 \end{vmatrix} = 0$$

b)

$$\begin{vmatrix} 1 & 2 & 2 & 1 \\ 2 & 1 & 1 & 2 \\ 2 & 1 & 1 & 2 \\ 1 & 2 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 2 & 1 \\ 0 & -3 & -3 & 0 \\ 0 & -3 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

c)

$$\begin{vmatrix} 1 & 1 & 1 & -1 \\ 1 & 1 & -1 & 1 \\ 1 & -1 & 1 & 1 \\ -1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & -1 \\ 0 & 0 & -2 & 2 \\ 0 & -2 & 0 & 2 \\ 0 & 2 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & -1 \\ 0 & -2 & -2 & 4 \\ 0 & -2 & 0 & 2 \\ 0 & 2 & 2 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 & -1 \\ 0 & -2 & -2 & 0 \\ 0 & 0 & 2 & -2 \\ 0 & 0 & 0 & 4 \end{vmatrix} = -16$$

Solution 1.3.8. We use determinants to check when A is invertible:

a) We find that A is invertible since

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 0 & 5 & 6 \\ 1 & 0 & 8 \end{vmatrix} = 1(40) + 1(-3) = 37 \neq 0$$

The inverse matrix is given by the cofactor matrix:

$$A^{-1} = \frac{1}{37} \begin{pmatrix} 40 & 6 & -5 \\ -16 & 5 & 2 \\ -3 & -6 & 5 \end{pmatrix}^T = \frac{1}{37} \begin{pmatrix} 40 & -16 & -3 \\ 6 & 5 & -6 \\ -5 & 2 & 5 \end{pmatrix}$$