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LECTURE 1

GRA 6035

EIVIND ERIKSEN

AUG 25, 2017

MATHEMATICS

Plan:

1. Introduction to the course
2. Linear systems and Gaussian elimination
3. Rank of a matrix

Reading:
[LS6E] or
[ME] 6.1, (6.2),
7.1-7.4, (7.5)
+
FORK 1003
Maths module

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1. Intro:

Topics:
* Matrix methods
* Optimization problems
* Differential / difference equations

Prep. course: FORK 1003 maths module

Lecture plan, workbook, other material: see link from It's L.

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2) Linear systems and Gaussian elimination.

Linear systems:

Defn: An $m \times n$ linear system is a system of m linear equations in n variables.

Ex:

$$\begin{aligned} x+y+z&=3 \\ x-y+z&=1 \\ x+2y+4z&=7 \end{aligned}$$

3×3 - linear system

In general, an equation in $x_1, x_2, \dots, x_n$ is linear if it can be written as

$$a_1x_1 + a_2x_2 + \dots + a_n \cdot x_n = b$$

where $a_1, a_2, \dots a_n, b$ are given numbers.

Ex:

$$\left.\begin{aligned} x_1^2 + x_2^3 &= 4 \\ x_1x_2 &= 1 \end{aligned}\right\} \text{ not linear}$$

$e^x = 4 - \ln(y)$ not linear

$\sqrt{2} + \ln(2)x - e^3y = 0$ is linear

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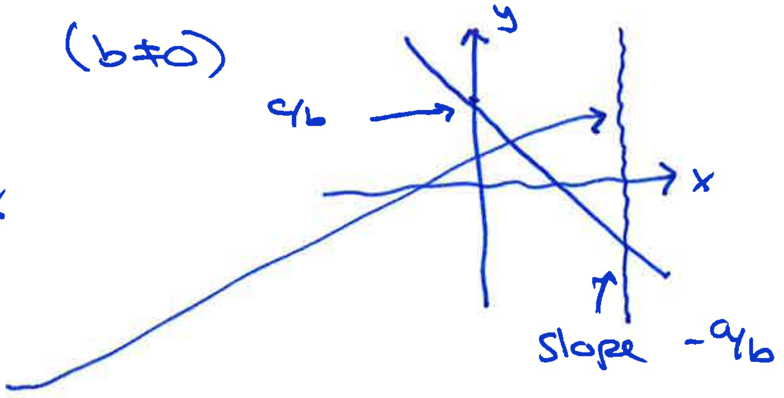
Case $n=2$: two variables

$ax + by = c$ (a, b, c given numbers)

$$\frac{by}{b} = \frac{c - ax}{b} \quad (b \neq 0)$$

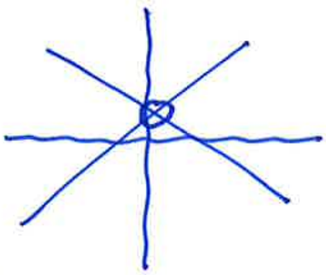
$$y = \frac{c}{b} - \frac{a}{b} \cdot x$$



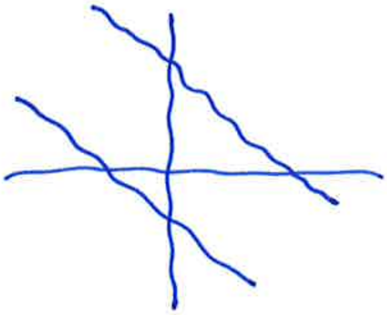
b=0: $ax = c$

$x = c/a$

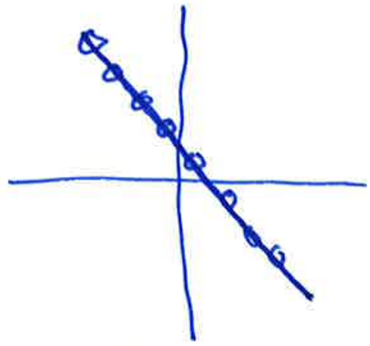
$$\boxed{\text{Linear equations in two variables} \iff \text{Straight lines in 2-dim space.}}$$



a) one solution



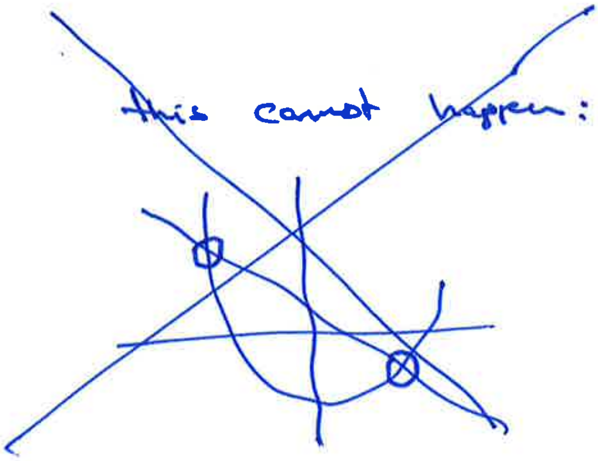
b) no solutions



c) infinitely many solutions

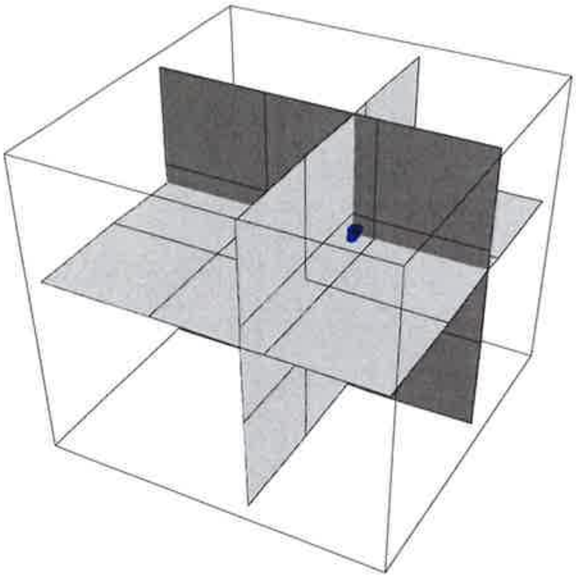
2x2 linear system

This cannot happen:

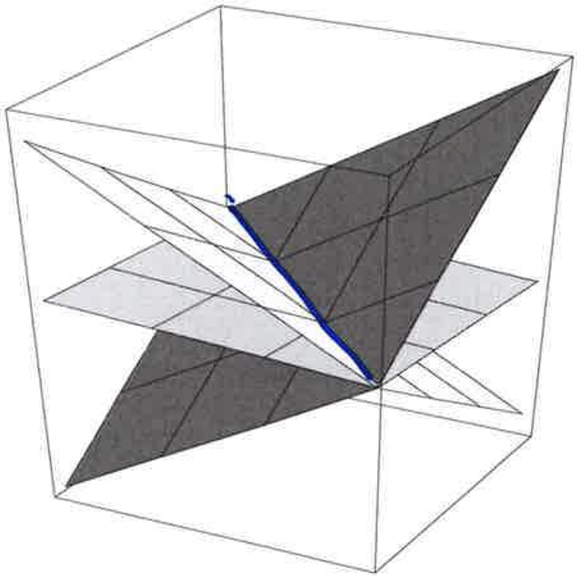


EXAMPLE: Three equations in three variables. Each equation determines a plane in 3-space.

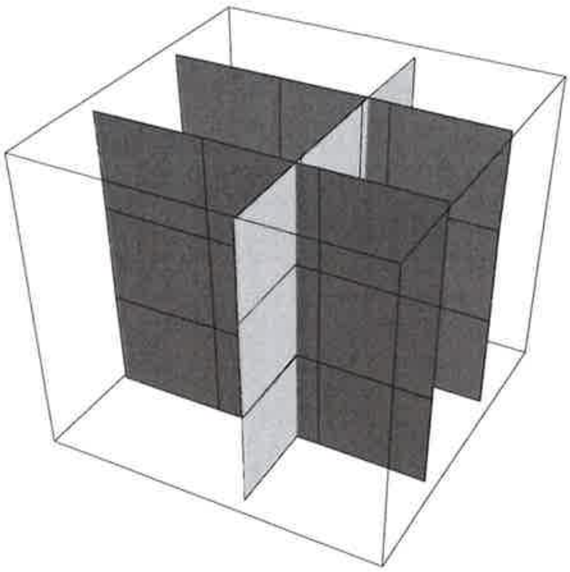
i) The planes intersect in one point. *(one solution)*



ii) The planes intersect in one line. *(infinitely many solutions)*



iii) There is not point in common to all three planes. *(no solution)*



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Theorem A:

Any $m \times n$ linear system has either

i) no solutions
ii) one unique solution
iii) infinitely many solutions

A linear system is called inconsistent if there are no solutions (i) and consistent otherwise (ii, iii); that is, if there are solutions.

The general linear system: $m \times n$ in $x_1, \dots, x_n$

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + \dots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + \dots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad\qquad\qquad &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + a_{m3}x_3 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

$a_{11}, \dots, a_{mn}$, $b_1, \dots, b_m$ are given constants

Coeff. matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

$m \times n$ - matrix

Augmented matrix

$$(A|\underline{b}) = \left(\begin{array}{ccc|c} a_{11} & \dots & a_{1n} & b_1 \\ a_{21} & \dots & a_{2n} & b_2 \\ \vdots & & & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{array}\right)$$

$m \times (n+1)$ - matrix

a_{23}: in eq. 2 in front of x_3

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Gaussian elimination:

Systematic, efficient, and pedagogical method for solving any linear system.

Method: Linear system

$$\begin{aligned} a_{11}x_1 + \dots + a_{1n}x_n &= b_1 \\ &\vdots \\ a_{m1}x_1 + \dots + a_{mn}x_n &= b_m \end{aligned}$$

i) Write down the augmented matrix of the system.

ii) Use elementary row operations to transform it into echelon form

iii) Find the solutions of the system corresponding to the echelon form.

Ex:

$$\begin{aligned} x+y+z &= 3 \\ x-y+z &= 1 \\ x+2y+4z &= 7 \end{aligned} \quad \rightsquigarrow \quad \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ 1 & 2 & 4 & 7 \end{array}\right)$$

augmented matrix.

Echelon form:

The first non-zero coeff. in a row is called a leading coefficient. A matrix is in echelon form if:

i) all zero rows are below all other rows (rows with only zeros)

ii) all coefficients under a leading coefficient are zero.

Typical echelon form:

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & 1 & -1 & 7 \\ 0 & 0 & 3 & 5 \end{array}\right) \rightarrow \begin{aligned} x+2y+3z &= 4 \\ y-z &= 7 \\ 3z &= 5 \end{aligned}$$

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Elementary row operations:

i) interchange two rows
ii) multiply row by $c \neq 0$
iii) add a multiple of one row to another row

$R(2) := R(2) - 1 \cdot R(1)$

$$(A|\underline{b}) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ 1 & 2 & 4 & 7 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & 0 & -2 \\ 1 & 2 & 4 & 7 \end{array}\right)$$

need these to be zero

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & 0 & -2 \\ 0 & 1 & 3 & 4 \end{array}\right) \begin{matrix} \\ \\ \tfrac{1}{2} \end{matrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 3 & 3 \end{array}\right) = E$$

echelon form

Find solutions:

- pivot = leading coefficient in the echelon form
- pivot positions = positions with pivots (in the echelon form)

$$x + y + z = 3$$
$$-2y = -2$$
$$3z = 3$$

back substitution

(3) $3z = 3 \Rightarrow \underline{z = 1}$

(2) $-2y = -2 \Rightarrow \underline{y = 1}$

(1) $x + y + z = 3$
$x + 1 + 1 = 3 \Rightarrow \underline{x = 1}$

Conclusion: One solution
$$\underline{\underline{(x, y, z) = (1, 1, 1)}}$$

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Theorem B:

a) Any linear system can be transformed into one on echelon form using elementary row operations. The echelon form is not unique, but the pivot positions are unique.

b) Any linear system can be transformed into one on reduced echelon form using elementary row operations, and the reduced echelon form is unique.

Reduced echelon form: echelon form +

a) all pivots are 1
b) all coeff. over a pivot are zero

Gauss-Jordan elimination:

Similar method, but end up with a reduced echelon form.

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Case: no solutions

Ex:

$$\left(\begin{array}{cccc|c} 1 & 3 & 4 & 7 & -1 \\ 0 & 0 & 1 & 3 & 5 \\ 0 & 0 & 0 & 0 & 4 \end{array}\right)$$

echelon form

$$\begin{aligned} x_1 + 3x_2 + 4x_3 + 7x_4 &= -1 \\ x_3 + 3x_4 &= 5 \\ 0 &= 4 \end{aligned}$$

no solutions

Fact: no solutions $\iff$ pivot position in last (augmented) column.

That is; the linear system is consistent $\iff$ no pivot position in the last column.

Degrees of freedom:

Assume no pivot position in last column.

Basic variables: with pivot position in the variable column
Free variables: without — " —

$$\begin{array}{cccc} x & y & z & w \end{array}$$

$$\left(\begin{array}{cccc|c} 1 & 4 & 0 & 7 & 1 \\ 0 & 1 & 3 & 4 & 0 \\ 0 & 0 & 0 & 2 & 7 \end{array}\right)$$

echelon form

no pivot positions: consistent system

x, y, w : basic
z : free

Degrees of freedom = # free variables $(=1)$

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$$\begin{cases} \text{no free variables} : \text{one solution} \\ \text{at least one degree of freedom} : \text{infinitely many solutions} \end{cases}$$

Ex:

$$\left(\begin{array}{cccc|c} 1 & 0 & 0 & 1 & 2 \\ 0 & 1 & 0 & 4 & 4 \\ 0 & 0 & 1 & 0 & 3 \end{array}\right)$$

$$\begin{aligned} \underline{x} \qquad\quad + w &= 2 \\ \underline{y} \quad + 4w &= 4 \\ \underline{z} \qquad &= 3 \end{aligned}$$

x, y, z : basic
w : free

infinitely many solutions
(one degree of freedom)

$$\begin{cases} z = \underline{3} \\ \underline{y} + 4w = 4 \\ \Rightarrow y = \underline{4 - 4w} \\ \underline{x} + w = 2 \\ \Rightarrow x = \underline{2 - w} \end{cases}$$

Solution:

$$\begin{aligned} x &= 2 - w \\ y &= 4 - 4w \\ z &= 3 \\ w &= w \quad \text{(free)} \end{aligned}$$

$\underline{w = 1}$: $(x, y, z, w) = (1, 0, 3, 1)$
$\underline{w = 2}$: $(x, y, z, w) = (0, -4, 3, 2)$
$\vdots$

one solution for each value of w
$\Rightarrow$ infinitely many solutions.

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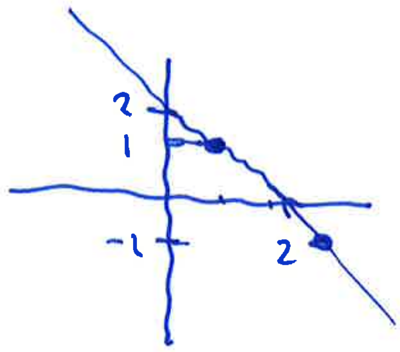

Easy example:

$$x + y = 2$$

$(1 \quad 1 \mid 2)$ $\quad \underline{x}+y=2$
echelon form

x basic $\qquad x = 2-y$
y free $\qquad y = y$ (free)

Note: Gaussian elimination says $\begin{cases} x \text{ basic} \\ y \text{ free} \end{cases}$

We could say $\begin{cases} x \text{ free} \\ y \text{ basic} \end{cases}$ $\qquad \begin{matrix} x \text{ free} \\ y = 2-x \end{matrix}$

$$\boxed{\#\text{ degrees of freedom} = \text{dimension of solution space}}$$

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3 Rank of a matrix

Defn: The rank of a matrix is the number of pivot positions in the echelon form of the matrix.

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 3 \end{pmatrix} = E \qquad rk(A) = 3$

$(A|\underline{b}) = \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ 1 & 2 & 4 & 7 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -2 & 0 & -2 \\ 0 & 0 & 3 & 3 \end{array}\right) = E$

$rk(A|\underline{b}) = 3$

Theorem C:
Consider an $m \times n$ linear system with coefficient matrix A and augmented matrix $(A|\underline{b})$. Then we have:

i) No solutions if and only if $rk(A|\underline{b}) > rk(A)$.

ii) Consistent if and only if $rk(A|\underline{b}) = rk(A)$.
In this case, we have

$$\# \text{ degrees of freedom} = n - rk(A)$$

In particular, in ii): Recall: $n = \#$ variables

$rk(A) = n$: one solution
$rk(A) < n$: infinitely many solutions

Result (A):

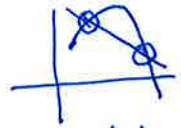
Any $m \times n$ linear system has either

i) one unique solution (consistent)

ii) no solutions (inconsistent)

iii) infinitely many solutions (consistent)

Non-linear case:



two solutions possible

Proof of A:

If there are two different solutions

$$(x_1, x_2, \dots, x_n) \neq (x_1', x_2', \dots, x_n')$$

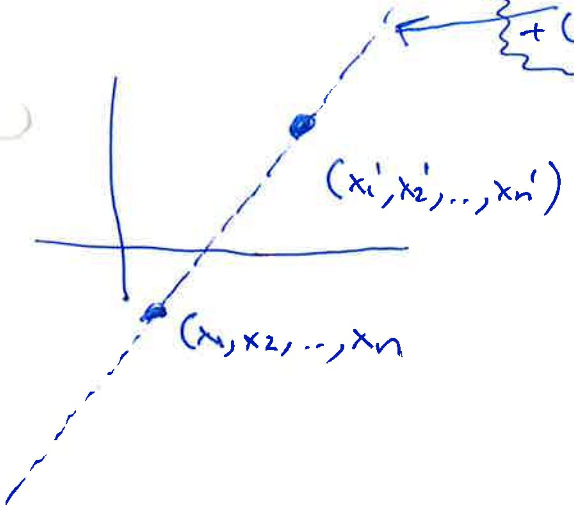
then

$$r(x_1, x_2, \dots, x_n) + (1-r)\cdot(x_1', x_2', \dots, x_n')$$
$$= (r x_1 + (1-r) x_1', \; r x_2 + (1-r) x_2', \; \dots, \; r x_n + (1-r) x_n')$$

is a solution for any number r. There are therefore infinitely many solutions

line through the two solutions.

$r(x_1, \dots x_n) + (1-r)\cdot(x_1', \dots, x_n')$



This ca be checked directly. Equation $\#i$ is satisfied since

$$a_{i1}\cdot(r x_1 + (1-r)x_1') + a_{i2}(r x_2 + (1-r) x_2') + \dots + a_{in}\cdot(r x_n + (1-r) x_n')$$
$$= r\cdot(a_{i1}x_1 + \dots + a_{in}x_n) + (1-r)\cdot(a_{i1}x_1' + \dots + a_{in}x_n')$$
$$= r\cdot b_i + (1-r) b_i = b_i$$

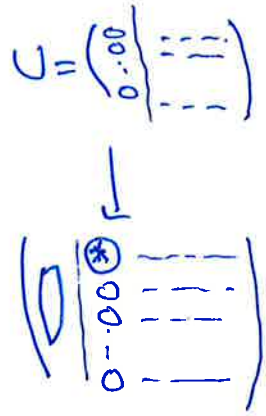
since $(x_1, \dots, x_n)$ and $(x_1', \dots, x_n')$ are solutions

So this equation holds for all i. Hence we have solution for all r. □

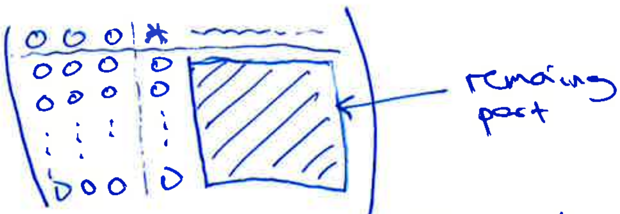
Proof of Result B and more details of Gauss/Gauss-Jordan elimination:

i) Any matrix can be reduced to an echelon form using Elementary row operations.

Start with any matrix U. Move to the right of any columns with only zeros, if any. Look at the first non-zero column, switch two rows in necessary to get a non-zero entry in the top corner. This is a pivot. Use it to get zeros under it.



Now look away from the first row, and look at the remaining part of the matrix.



Repeat the steps above. Since the new matrix is smaller than the original (one row less), we sooner or later get an echelon form this way.

Multiply each row with pivot to set pivot = 1. Use each pivot to set zeros over it, starting from the rightmost. You get a reduced echelon form.

ii) If a matrix A can be reduced to reduced echelon forms U, V using elementary row operations, $U = V$.

We have $(A \vdots 0) \dashrightarrow (U \vdots 0)$ and $(A \vdots 0) \dashrightarrow (V \vdots 0)$, and elementary row operations do not change solutions of linear systems. So $U \cdot \underline{x} = \underline{0}$ and $V \cdot \underline{x} = \underline{0}$ have the same solutions.

Write $U = (c_1 | c_2 | \dots | c_n)$ and $V = (c_1' | c_2' | \dots | c_n')$ in terms of their columns. We have

$$c_i = x_1 c_1 + x_2 c_2 + \dots + x_{i-1} c_{i-1} \Leftrightarrow c_i \text{ non-pivot column in } U$$

$$\Updownarrow$$

$$c_i' = x_1 c_1' + x_2 c_2' + \dots + x_{i-1} c_{i-1}' \Leftrightarrow c_i' \text{ non-pivot column in } V$$

So U and V have the same pivot columns; They are in the positions, and the pivot colums are

$$\begin{array}{c} 1 \\ \\ \\ \\ \\ \end{array}\begin{pmatrix}1\\0\\0\\ \vdots\\0\end{pmatrix}, \quad \begin{array}{c} \\ 2 \\ \\ \\ \\ \end{array}\begin{pmatrix}0\\1\\0\\ \vdots\\0\end{pmatrix}, \quad \begin{array}{c} \\ \\ 3 \\ \\ \\ \end{array}\begin{pmatrix}0\\0\\1\\ \vdots\\0\end{pmatrix}, \ \ldots, \quad \begin{array}{c} \\ \\ \\ \\ r \\ \\ \\ \end{array}\begin{pmatrix}0\\0\\0\\ \vdots\\1\\ \vdots\\0\end{pmatrix}$$

To show that $U=V$, we must show that non-pivot columns are equal. But each non-pivot colum satisfy

$$\begin{cases} C_i = x_1 C_1 + \cdots + \\ C_i' = x_1 C_1' + \cdots + \end{cases}$$ (linear combination of pivot colums to the right)

and the pivot colums are equal, and the coeff's x_i are equal.

Hence $U=V$.

iii) You can always get from an echelon matrix to a reduced echelon matrix, using elementary row operations, without changing the pivot positions.

This follows from the last steps in i).

$\square$

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Proof of Thm C: $m \times n$ linear system $\rightarrow$ n variables

no solutions $\Leftrightarrow$ pivot position in last column $\Leftrightarrow$ $rk(A|\underline{b}) = rk(A) + 1$

$> rk(A)$

Consistent: $\Leftrightarrow$ no pivot position in last column $\Leftrightarrow$ $rk(A|b) = rk(A)$.

If it is consistent: n variables

$rk(A)$ pivots $= rk(A)$ basic variables

$\Downarrow$

$n - rk(A)$ free variables