LECTURE 11

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NOV 10, 2017

GRA 6035 MATHEMATICS

BI

Plan:

Review: First order differential equations

1. Stability and equilibrium states
2. Second order differential equation
3. Systems of differential equations

Reading:

[ME] 24.1-24.3, 25.1-25.2, 24.5, 25.4

[ODE] Diff. Eq. 1.8-1.10, 2.1-2.3

Reminders:

Plenary Session Mon 1700.

No Office hours Thu.

New version Workbook: 3 Excel problems (Lecture 10-11)

— " — [ODE] Diff. equations (new Ch 2.1-2.3)

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Review: First order differential equations

$$y' = F(t,y)$$

1) Linear: $y' + a(t)y = b(t)$

Int. factor: $u = e^{\int a(t)dt} \Rightarrow (u\cdot y)' = b(t)\cdot u(t) \Rightarrow \boxed{y = \frac{1}{u}\int b(t)u(t)\,dt}$

2) Separable: $y' = f(y)\cdot g(t)$

$$\Rightarrow \frac{1}{f(y)}y' = g(t) \Rightarrow \boxed{\int \frac{1}{f(y)}dy = \int g(t)\,dt}$$

3) Exact: $h'_t + h'_y\cdot y' = 0$ for $h = h(t,y)$

$$\Rightarrow \boxed{h(t,y) = C}$$

Ex: $y' + 2ty = 2t$, $y(0) = 2 \leftrightarrow \begin{cases} t=0 \\ y=2 \end{cases}$

Linear:

$\int 2t\,dt = t^2 + c$

$u = e^{t^2}$

$(e^{t^2}\cdot y)' = 2t\cdot e^{t^2}$

$e^{t^2}\cdot y = \int 2t\cdot e^{t^2}dt = e^{t^2} + C$

$\underline{\underline{y = 1 + C\cdot e^{-t^2}}}$

$y(0)=2$: $2 = 1 + C\cdot e^0$

$2 = 1 + C$

$\underline{C = 1}$

$y(t) = \underline{\underline{1 + e^{-t^2}}}$

Separable: $y' = 2t - 2ty$

$= 2t\cdot(1-y)$

$y' = (1-y)\cdot 2t$

$\frac{1}{1-y}y' = 2t$

$\int \frac{1}{1-y}dy = \int 2t\,dt$

$-\ln|1-y| = t^2 + C$

$\ln|1-y| = -t^2 - C$

$|1-y| = e^{-t^2}\cdot e^{-C}$

$1-y = \pm e^{C}\cdot e^{-t^2} = K\cdot e^{-t^2}$

$\underline{\underline{y = 1 - K\cdot e^{-t^2}}}$

① Equilibrium states and stability.

$y' = F(y)$ autonomous case (no t in the diff. eqn)

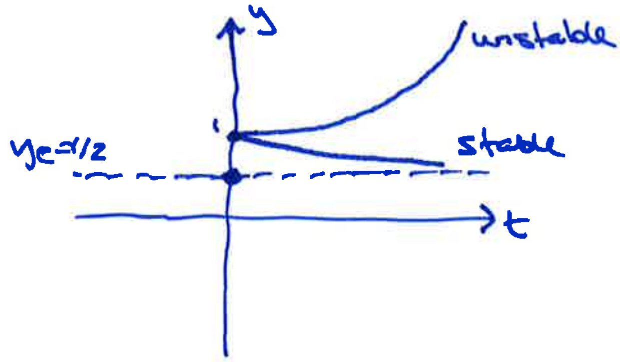
Ex: $y' + 2y = 1 \Rightarrow y' = 1 - 2y$

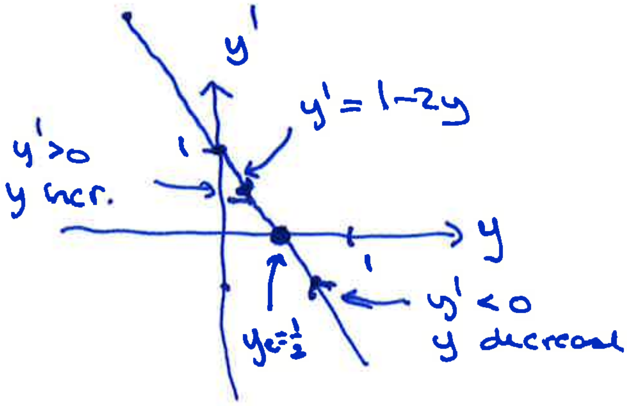
Equilibrium states: $y = y_e$ is equilibrium state if $F(y_e) = 0$
(steady state)

$y' = 1 - 2y$: $1 - 2y = 0$
$y = 1/2$
$y_e = 1/2$

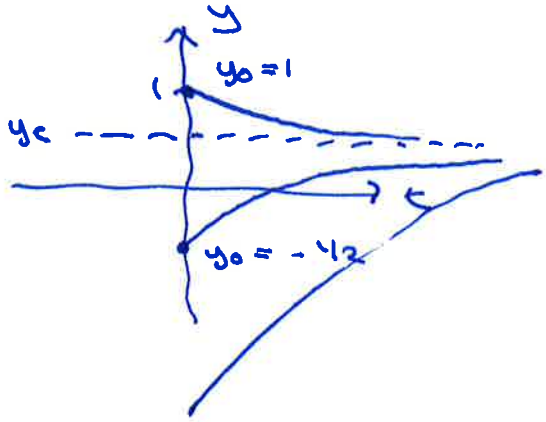
If you start at y_e, you will stay there as time passes.

What if $y_0 \neq y_e$ but close to y_e?





phase diagram



(globally asympt.)

This means that $y_e = 1/2$ is stable equilibrium state.

Defn: y_e is stable if $y_0 \neq y_e$ but close to y_e
$\Rightarrow y(t)$ approaches y_e as time passes

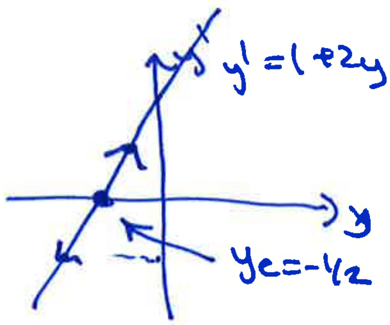
" unstable otherwise

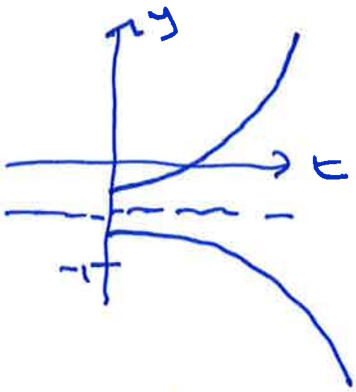
y_e is globally asymptotically stable if $y_0 \neq y_e \Rightarrow y(t)$ approaches y_e

Ex: $y' = 1 + 2y$

Eq. state: $1 + 2y = 0$

$y_e = -1/2$





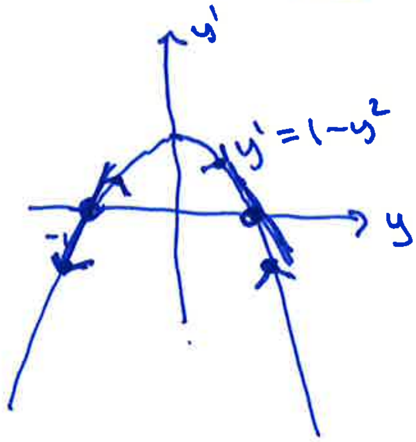
unstable

$y' = 1 - y^2$

Eq. state: $1 - y^2 = 0$

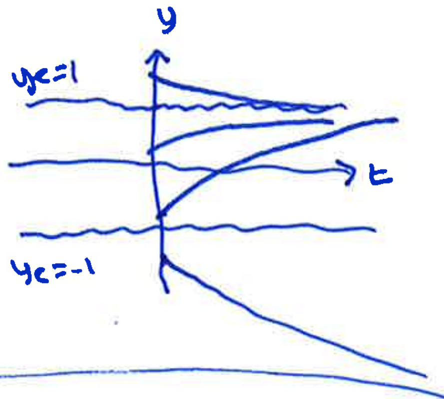
$y = \pm 1$

$y_e = -1$, $y_e = 1$



$y_e = 1$: stable, not globally asymp. stable

$y_e = -1$: unstable



Result: $y' = F(y)$ autonomous, $y = y_e$ eq. state

$F'(y_e) > 0$: y_e unstable

$F'(y_e) < 0$: y_e stable

Ex: $y' = 1 - y^2$ $\quad F(y) = 1 - y^2 \quad F(y) = 0$: $1 - y^2 = 0$

$y = \pm 1$

$F'(y) = -2y$

$F'(1) = -2 \Rightarrow y_e = 1$ stable

$F'(-1) = 2 \qquad y_e = -1$ unstable

2 Second order differential equations

Ex: $y'' = 6$

$y' = \int 6\, dt = 6t + C$

$y = \int 6t + C\, dt = 6 \cdot \frac{1}{2}t^2 + C \cdot t + D$

$$\underline{\underline{y = 3t^2 + Ct + D}}$$

general solution

Second order diff.-eqn:

- the general solution will depend on two undetermined coeff.
- diff. eqn + two initial conditions $\Rightarrow$ unique solution.

Ex: $y'' = 6$, $y(0) = 1$, $y'(0) = 0$

$y' = 6t + C$ $\quad$ $y'(0) = 0$: $\quad 0 = C \quad \underline{\underline{C = 0}}$

$y = 3t^2 + Ct + D$ $\quad$ $y(0) = 1$: $\quad 1 = D \Rightarrow \underline{\underline{D = 1}}$

$$y(t) = \underline{\underline{3t^2 + 1}}$$

Linear second order diff. eqn:

Defn: Linear means that it can be written as

$$y'' + a(t)y' + b(t)y = f(t)$$

It is called homogeneous if $f(t)=0$, otherwise inhomogeneous.

Ex: $y'' + 7y' + 10y = 2$

Superposition principle:

A linear second order diff. equation

$$y'' + a(t)y' + b(t)y = f(t)$$

has general solution

$$y = y_h + y_p$$

where

y_h = general solution of the homogeneous diff. eqn.

$$y'' + a(t)y' + b(t)y = 0$$

y_p = a particular solution of

$$y'' + a(t)y' + b(t)y = f(t)$$

Comment:

1. $D(y)$ = left-hand side

$D(y_1 + y_2) = D(y_1) + D(y_2)$

$D(c \cdot y_1) = c \cdot D(y_1)$

2. works also for first order linear diff. eqn.

Ex: $y'' + 7y' + 10y = 2$

$\Downarrow$

$y = \underline{y_h} + y_p = \underline{\underline{C_1 e^{-2t} + C_2 e^{-5t} + 1/5}}$

Characteristic equation

$\parallel$

$\underline{y_h}$: $y'' + 7y' + 10y = 0$

$r^2 + 7r + 10 = 0 \longleftarrow$ $\underline{\text{char. eqn}}$: $\begin{cases} y'' \rightsquigarrow r^2 \\ y' \rightsquigarrow r \\ y \rightsquigarrow 1 \end{cases}$

$r = \frac{-7 \pm \sqrt{49 - 4 \cdot 10}}{2} \rightarrow r_1 = -2,\ r_2 = -5$

$\Downarrow$

$y_h = \underline{C_1 \cdot e^{-2t} + C_2 \cdot e^{-5t}}$

$\underline{y_p}$: $y'' + 7y' + 10y = 2$ try $\begin{cases} y = A \\ y' = 0 \\ y'' = 0 \end{cases}$

$0 + 7 \cdot 0 + 10 \cdot A = 2 \longleftarrow$

$A = 2/10 = \underline{1/5}$

$y_p = \underline{1/5}$

General methods:

a) Homogeneous case: y_h

~~?~~ Diff. eqn.: $y'' + a(t)y' + b(t)y = 0$

Special case: constant coeff. $y'' + ay' + by = 0$

Char. eqn: $r^2 + ar + b = 0$

$$r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

if $a(t), b(t)$ are not constants, we cannot use char. eqn.

General solution:

i) $a^2 - 4b > 0$:

$$\left.\begin{array}{l} r_1 = \dfrac{-a + \sqrt{a^2 - 4b}}{2} \\ r_2 = \dfrac{-a - \sqrt{a^2 - 4b}}{2} \end{array}\right\} r_1 \neq r_2 \Rightarrow y = C_1 \cdot e^{r_1 t} + C_2 e^{r_2 t}$$

ii) $a^2 - 4b = 0$:

$$r_1 = r_2 = -\frac{a}{2} \Rightarrow y = C_1 \cdot e^{r_1 t} + C_2 t e^{r_1 t} = (C_1 + C_2 t) e^{r_1 t}$$

iii) $a^2 - 4b < 0$:

$$r = \frac{-a \pm \sqrt{4b - a^2} \cdot \sqrt{-1}}{2}$$

$$\alpha = -\frac{a}{2} \quad \beta = \frac{\sqrt{4b - a^2}}{2} \Rightarrow y = e^{\alpha t} \cdot (C_1 \cdot \cos \beta t + C_2 \cdot \sin \beta t)$$

Explanation: $y'' + ay' + by = 0$

Assume $y = e^{rt}$ is a solution for a number r

$$y = e^{rt}, \quad y' = r \cdot e^{rt}, \quad y'' = r^2 \cdot e^{rt}$$

$$(r^2 e^{rt}) + a \cdot (r e^{rt}) + b \cdot (e^{rt}) = 0$$

$$(r^2 + ar + b) e^{rt} = 0$$

$$\underline{r^2 + ar + b = 0}$$

Concl: $y = e^{rt}$ solution $\Longleftrightarrow$ $r^2 + ar + b = 0$ (char. eqn.)

b) How to find y_p: Method of undetermined coefficients.

Ex: $y'' + 7y' + 12y = t$

$$y = y_h + y_p = C_1 e^{-3t} + C_2 e^{-4t} + y_p$$

y_h: $y'' + 7y' + 12y = 0$ $\Rightarrow$ $y_h = C_1 \cdot e^{-3t} + C_2 e^{-4t}$

$$r^2 + 7r + 12 = 0$$

$$r = -3, -4$$

$$r = \frac{-7 \pm \sqrt{49 - 4 \cdot 12}}{2}$$

A, B: const are undet. coeff.

y_p: $y'' + 7y' + 12y = t$ $\rightarrow$ $\cancel{f(t) = t, \; f'(t) = 1, \; f''(t) = 0}$ $\Rightarrow$ $y = At + B$

$$y' = A, \quad y'' = 0$$

$$0 + 7 \cdot A + 12(At + B) = t$$

$$\underbrace{(12A)}_{=1} t + \underbrace{(12B + 7A)}_{=0} = t$$

$12A = 1$ $\qquad$ $A = \frac{1}{12}$

$12B + 7A = 0$ $\qquad$ $12B = -\frac{7}{12}$

$$B = -\frac{7}{144}$$

$$\underline{y_p = \frac{1}{12} t - \frac{7}{144}}$$

Note: If the initial guess for y does not work, try to multiply it by t.

4 Systems of differential equations

Ex: $$y_1' = y_1 - 2y_2$$
$$y_2' = -2y_1 + y_2$$

$y_1 = y_1(t)$
$y_2 = y_2(t)$

coupled diff. eqn.

Syst in $y_1(t), y_2(t), \dots, y_n(t)$:

$$y_1' = F_1(t, y_1, \dots, y_n)$$
$$y_2' = F_2(t, y_1, \dots, y_n)$$
$$\vdots$$
$$y_n' = F_n(t, y_1, \dots, y_n)$$

Special case: $n=2$ linear system of diff. eqn's

$$y_1' = a_{11} y_1 + a_{12} y_2$$
$$y_2' = a_{21} y_1 + a_{22} y_2$$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\boxed{\underline{y}' = A \cdot \underline{y}}$$

Ex: $$\underline{y}' = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \cdot \underline{y}$$

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If A is diagonal:

Ex: $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$\Updownarrow$

$$\left.\begin{aligned} y_1' &= 3y_1 \\ y_2' &= -y_2 \end{aligned}\right\} \text{ decoupled system}$$

$y_1' = 3y_1$ $\qquad$ $y_2' = -y_2$

$y_1 = c_1 e^{3t}$ $\qquad$ $y_2 = c_2 \cdot e^{-t}$

$\Downarrow$

General solution: $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} c_1 e^{3t} \\ c_2 e^{-t} \end{pmatrix}$

What if A is not diagonal?

Try to diagonalize A

$\Updownarrow$

Method to solve the system using eigenvalues and eigenvectors of A if A is diagonalizable.

$$\underline{\underline{x = c_1 \cdot \underline{v_1} e^{\lambda_1 t} + c_2 \cdot \underline{v_2} e^{\lambda_2 t} + \dots + c_n \cdot \underline{v_n} e^{\lambda_n t}}}$$

general solution when $\lambda_1, \dots, \lambda_n$ eigenvalues of A and $\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}$ linearly independent eigenvectors of A.

Explanation:

Linear systems of diff. eqn's can be written in the form

$$\begin{aligned} y_1' &= a_{11} y_1 + a_{12} y_2 + \dots + a_{1n} y_n \\ y_2' &= a_{21} y_1 + a_{22} y_2 + \dots + a_{2n} y_n \\ \vdots \quad & \quad \vdots \\ y_n' &= a_{n1} y_1 + a_{n2} y_2 + \dots + a_{nn} y_n \end{aligned} \iff \underline{y}' = A\underline{y}$$

Solution method in case A is diagonalizable:

Eigenvalues of A: $\lambda_1 \dots \lambda_n$

Eigenvector of A: $\underline{v}_1, \dots, \underline{v}_n$

(lin indep. such that $A\underline{v}_i = \lambda_i \underline{v}_i$)

$$\Rightarrow P = (\underline{v_1} | \underline{v_2} | \dots | \underline{v_n}) \quad D = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \; \lambda_n \end{pmatrix}$$

$$P^{-1}AP = D \iff A = PDP^{-1}$$

Rewrite system $\underline{y}' = A\underline{y}$ using $\boxed{\underline{w} = P^{-1}\underline{y} \iff \underline{y} = P \cdot \underline{w}}$

$$\left.\begin{aligned} \underline{y}' &= (P \cdot \underline{w})' = P \cdot \underline{w}' \\ A\underline{y} &= (PDP^{-1})\underline{y} = PD \cdot \underline{w} \end{aligned}\right\} \underline{y}' = A\underline{y} \iff P\underline{w}' = PD\underline{w} \iff \underline{w}' = D\underline{w}$$

decoupled system:

$$\begin{aligned} w_1' &= \lambda_1 w_1 \\ w_2' &= \lambda_2 w_2 \\ &\vdots \\ w_n' &= \lambda_n w_n \end{aligned}$$

$$\Updownarrow$$

$$\begin{aligned} w_1 &= c_1 \cdot e^{\lambda_1 t} \\ w_2 &= c_2 \cdot e^{\lambda_2 t} \\ &\vdots \\ w_n &= c_n \cdot e^{\lambda_n t} \end{aligned}$$

Solution for $\underline{w}$

$$\underline{y} = P \cdot \underline{w} = (\underline{v_1} | \underline{v_2} | \dots | \underline{v_n}) \cdot \begin{pmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \\ \vdots \\ c_n e^{\lambda_n t} \end{pmatrix}$$

$$= \underline{v}_1 \cdot c_1 e^{\lambda_1 t} + \underline{v}_2 \cdot c_2 e^{\lambda_2 t} + \dots + \underline{v}_n \cdot c_n e^{\lambda_n t}$$

$$\boxed{\underline{y} = c_1 \underline{v}_1 e^{\lambda_1 t} + c_2 \underline{v}_2 e^{\lambda_2 t} + \dots + c_n \underline{v}_n e^{\lambda_n t}}$$

Ex: $\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ $\Leftrightarrow$ $\begin{aligned} y_1' &= y_1 - 2y_2 \\ y_2' &= -2y_1 + y_2 \end{aligned}$

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$A = \begin{pmatrix} 1 & -2 \\ -2 & 1 \end{pmatrix}$ A is symmetric and therefore diagonalizable.

Eigenvalues: $\begin{vmatrix} 1-\lambda & -2 \\ -2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 = \lambda^2 - 2\lambda - 3 = 0$

$\lambda_1 = 3, \lambda_2 = -1$

Eigenvectors:

$\lambda = 3$: $\begin{pmatrix} -2 & -2 \\ -2 & -2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad -2y_1 - 2y_2 = 0$

$\Updownarrow$

$y_1 = -y_2$

$y_2 = \text{free}$

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -y_2 \\ y_2 \end{pmatrix} = y_2 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\Updownarrow$

$\underline{v_1} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda = -1$: $\begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad 2y_1 - 2y_2 = 0$

$\Updownarrow$

$y_1 = y_2$

$y_2 = \text{free}$

$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} y_2 \\ y_2 \end{pmatrix} = y_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\Updownarrow$

$\underline{v_2} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

General solution:

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 \cdot \underline{v_1} e^{\lambda_1 t} + C_2 \underline{v_2} e^{\lambda_2 t} = C_1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{3t} + C_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} -C_1 e^{3t} + C_2 e^{-t} \\ C_1 e^{3t} + C_2 e^{-t} \end{pmatrix}$$