LECTURE 13

GRA 6035

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Plan:

(1) Review: Important topics

(2) Final exam 12/2016

(1) Main topics

* matrix methods
* unconstrained optimization
* constrained optimization
* differential (difference) equations

② Final exam 12/2016

BI

Question I:

$$A = \begin{pmatrix} -2 & 2 & 0 \\ -1 & 0 & 2 \\ 0 & -1 & 2 \end{pmatrix}$$

a)

$$|A| = -2 \cdot 2 - 2 \cdot (-2) = -4 + 4 = \underline{\underline{0}}$$

$rk A = \underline{\underline{2}}$ since $|A| = 0 \Rightarrow rk A < 3$

and

$$\begin{vmatrix} -2 & 2 \\ -1 & 0 \end{vmatrix} = 2 \neq 0 \Rightarrow rk A = 2$$

b)

$$\left(\begin{array}{ccc|c} -2 & 2 & 0 & 0 \\ -1 & 0 & 2 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ -2 & 2 & 0 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right) \,]\, -2$$

$$\to \left(\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 0 & 2 & -4 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} -1 & 0 & 2 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$-x + 2z = 0$

$-y + 2z = 0$

$\underline{z \text{ free}}$

$x = 2z$

$y = 2z$

$z = z$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2z \\ 2z \\ z \end{pmatrix} = \underline{z \cdot \begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}}$$

Solution: $\text{span}(\underline{v_1})$ with $\underline{v_1} = \underline{\underline{\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}}}$

$\shortparallel$

$\{ r \cdot \underline{v_1} : r \text{ number} \}$

c) $\begin{vmatrix} -2-\lambda & 2 & 0 \\ -1 & 0-\lambda & 2 \\ 0 & -1 & 2-\lambda \end{vmatrix} = 0 \iff |A-\lambda I| = 0$

$$(-2-\lambda)\cdot\left(-\lambda(2-\lambda)+2\right) - 2\cdot\left(-1(2-\lambda)\right) = 0$$

$$\underline{(-2-\lambda)\cdot(\lambda^2-2\lambda+2)} \; \underline{+2(2-\lambda)}$$

$$\lambda^3 \cancel{-2\lambda^2} \cancel{+2\lambda^2} \cancel{-2\lambda} \cancel{+4\lambda} \cancel{-4} \cancel{+4} \cancel{-2\lambda} = 0$$

$$\lambda^3 + \cancel{2} = 0$$

$\lambda^3 = 0$ $\quad$ $\underline{\underline{\lambda = 0}}$ has multiplicity 3

det(A) = 0
$\Updownarrow$
$\lambda = 0$ is a solution
$\Updownarrow$
no const. in the char. eqn.

Topic: Matrix methods

* linear system (Gaussian elimination
* determinants (minors and rank
* linear independence of vectors
* eigenvalues and eigenvectors, diagonalization, Markov chains
* definiteness of symmetric matrices, (leading) principal minors

Reduced rank criterion:

A symmetric $n \times n$-matrix with $rk(A) = r < n$

$D_1 > 0,\ D_2 > 0, \ldots, D_r > 0,\ D_{r+1} = \ldots = D_n = 0 \Rightarrow A$ pos. semidefin.

$D_1 < 0,\ D_2 > 0, \ldots, (-1)^r \cdot D_r > 0,\ D_{r+1} = \ldots = D_n = 0 \Rightarrow A$ neg. semidefin.

Ex:

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{array}{l} D_1 = 1 \\ D_2 = 1 \\ D_3 = 1 \cdot (1-1) = 0 \end{array}$$

$$\downarrow$$

$rk A = \underline{2}$ since

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

RRC: A is <u>positive semidefinite</u> since $D_1 = 1, D_2 = 1, D_3 = 0, D_4 = 0$ and $rk A = \underline{2}$.

Alternative:

$\Delta_1 = 1, 1, 1, 1$

$\Delta_2 = 1, 0, 1, 1, 1, 0$

$\Delta_3 = 0, 0, 0, 0$

$\Delta_4 = 0$

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2

a) $y_{t+2} = 3y_{t+1} - 2y_t$, $y_0 = 1$, $y_1 = 2$

$y_{t+2} - 3y_{t+1} + 2y_t = 0$ second order linear homogeneous

$y_t = y_t^h = \underline{C_1 + C_2 \cdot 2^t}$ general solution

y_t^h: $r^2 - 3r + 2 = 0$

$\underline{r=1}$, $\underline{r=2}$ $\longrightarrow y_t^h = C_1 \cdot 1^t + C_2 \cdot 2^t$

<u>remember</u>: r^t <u>not</u> e^{rt} $= C_1 + C_2 \cdot 2^t$

<u>Particular solution</u>: $y_t = C_1 + C_2 \cdot 2^t = \underline{\underline{2^t}}$

$y_0 = 1$: $1 = C_1 + C_2 \cdot 2^0 = C_1 + C_2$

$y_1 = 2$: $-\ 2 = C_1 + C_2 \cdot 2^1 = C_1 + 2C_2$

$= -1 \qquad = -C_2$

$\underline{C_2 = 1}$, $\underline{C_1 = 0}$

b) $y' - y\cdot \ln t = y$

$$y' + \underbrace{(-\ln t - 1)}_{a(t)}\, y = 0$$

$$\left(y\cdot e^{-t\ln t}\right)' = 0$$

$$y\cdot e^{-t\ln t} = C$$

$$y = \underline{\underline{C\cdot e^{t\ln t}}}$$

Alt:

$$y' = y + y\cdot \ln t$$

$$y' = y\cdot(1+\ln t)$$

$$\frac{1}{y}\,y' = 1 + \ln t$$

$$\int \frac{1}{y}\,dy = \int 1 + \ln t\,dt$$

$$\ln|y| = t\ln t + C$$

$$y = \pm\, e^{t\ln t + C}$$

$$= \underline{K\cdot e^{t\ln t}}$$

first order differential eqn.

linear

$$\int a(t)dt = \int -\ln t - 1\,dt$$

$$= -(t\ln t - t) - t + C$$

$$= -t\cdot\ln t + t - t + C$$

$$= -t\cdot\ln t + C$$

$$\underline{u = e^{-t\ln t}} = \left(e^{\ln t}\right)^{-t}$$

$$= \underline{t^{-t}}$$

Alt:

$$\int -\ln t - 1\,dt =$$

$$\int -1\cdot(\ln t + 1)\,dt$$

$$\boxed{\begin{array}{ll} u = -t & v = \ln t + 1 \\ u' = -1 & v' = 1/t \end{array}}$$

$$= -t\cdot(\ln t + 1) - \int -t\cdot\frac{1}{t}\,dt$$

$$= -t\cdot\ln t - t + \int 1\,dt$$

$$= -t\cdot\ln t - \cancel{t} + \cancel{t} + C$$

$$= -t\cdot\ln t + C$$

c) $y e^{ty} + t e^{ty} y' = 1$, $y(1) = \ln(2)$

first order diff. eqn.

exact?

$$\underbrace{(y e^{ty} - 1)}_{h'_t} + \underbrace{(t e^{ty})}_{h'_y} y' = 0$$

Try to find h with these properties.

$$h'_t = y e^{ty} - 1$$

$$h = \int y e^{ty} - 1 \, dt = \int y e^{ty} dt - t$$

$u = ty$

$du = u'_t \cdot dt$

$= y \cdot dt$

$$= \int \cancel{y} \cdot e^u \cdot \frac{du}{\cancel{y}} - t$$

$$= \int e^u du - t = e^u - t + C(y) = \underline{e^{ty} - t + C(y)}$$

$\Downarrow$

$$h'_y = \underline{e^{ty} \cdot t} - 0 + C'(y) = \underline{t e^{ty}}$$

$C'(y) = 0$

can choose $C(y) = 0$

Conclusion. The eqn. is exact and

$$h = \underline{e^{ty} - t = C}$$ implicit general solution

$e^{ty} = t + C$

$ty = \ln(t + C)$

$$y = \frac{\ln(t+C)}{t}$$

$y(1) = \ln 2$: $\ln 2 = \frac{\ln(1+C)}{1}$

$C = 1$

$$y = \frac{\ln(t+1)}{t}$$

Topic: Differential / difference equation

* first order differential eqn.: linear, separable, exact
* second order linear differential / difference eqn.
* new: linear systems (2x2 systems when A is diagonalizable)
* new: equilibrium states and stability.

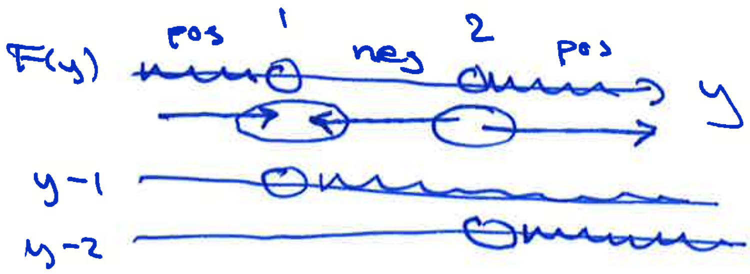
Ex: Mock exam, pb. 2b):

$$y' = y^2 - 3y + 2$$

Eq. states: $F(y) = y^2 - 3y + 2 = 0$

$y_e = 1$, $y_e = 2$

Stability: $F(y) = y^2 - 3y + 2 = (y-1)(y-2)$



y_0 close to 1
⇓
$y(t) \to 1$
stable

$y_e = 1$ is not globally as. stable since $y_0 > 2$ ⇓ $y(t)$ moves away.

y_0 close to 2
⇓
$y(t)$ will move away
unstable

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Question 3.

$$f(x,y,z) = \underline{5x^2 - 8xy - 4xz + 5y^2 - 4yz + 8z^2} + 1$$
$$= Q(x,y,z) + 1$$

a) Is f convex?

$$A = \begin{pmatrix} 5 & -4 & -2 \\ -4 & 5 & -2 \\ -2 & -2 & 8 \end{pmatrix} \qquad \left(H(f) = 2 \cdot A \right)$$

$D_1 = \underline{5}$

$D_2 = 25 - 16 = \underline{9}$

$D_3 = -2(8+10) + 2\cdot(-10-8) + 8 \cdot 9$

$= -36 - 36 + 72 = \underline{0}$

$\operatorname{rk} A = 2$ since $|A| = 0$, $D_2 \neq 0$

$\Downarrow$ ERC

A pos. semidefn. $\Rightarrow$ $\underline{\underline{f \text{ convex}}}$

b) Stationary pts:

$$\text{FOC:} \begin{cases} f'_x = 10x - 8y - 4z = 0 \\ f'_y = -8x + 10y - 4z = 0 \\ f'_z = -4x - 4y + 16z = 0 \end{cases}$$

$$\left(\begin{array}{ccc|c} 10 & -8 & -4 & 0 \\ -8 & 10 & -4 & 0 \\ -4 & -4 & 16 & 0 \end{array}\right) \xrightarrow[\text{steps}]{} \left(\begin{array}{ccc|c} 1 & 1 & -4 & 0 \\ 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right) \qquad \begin{aligned} x &= 2z \\ y &= 2z \\ z &= z \end{aligned}$$

$\underline{z \text{ free}}$

Stationary pts: $(x,y,z) = \underline{\underline{(2z, 2z, z)}}$ with z free

c) $g(x,y,z) = w \cdot \ln(w)$, $\quad w = f(x,y,z)$

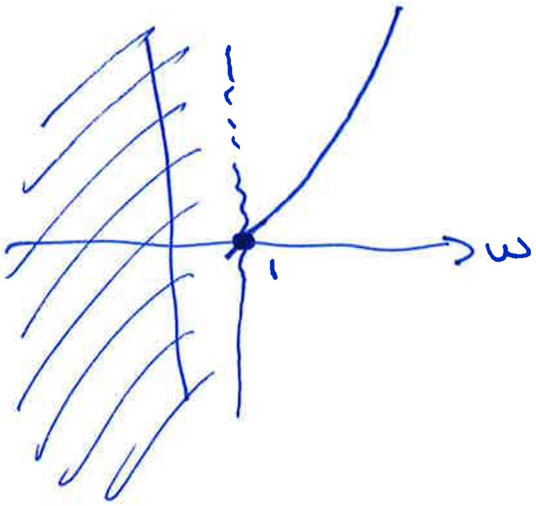
Find min g:

(*) $w = f(x,y,z)$: f convex, $(2z,2z,z)$ stationary pts $\Big\}$ $f_{min} = f(0,0,0) = \underline{1}$

$$\Downarrow$$

$$w = f(x,y,z) \geq 1$$

(*) $g(w) = w \cdot \ln(w)$, $\quad w \geq 1$



$g(1) = 1 \cdot \ln(1) = 0$

$g'(w) = 1 \cdot \ln(w) + w \cdot \frac{1}{w}$

$= \ln(w) + 1 > 0$

$$\Downarrow$$

$$\min g = g(1) = \underline{\underline{0}}$$

Topic: Unconstrained optimization

* stationary pts
* convex / concave fn $\Rightarrow$ stationary pts are global min / max
* second derivative test $\Rightarrow$ classify stationary pts as local max / min, or saddle pts.

(from Pb. 3)

4. min $f(x,y,z)$ wh $x+y-4z=8$

Topic:
Constrained opt.
* Lagrange
* Kuhn-Tucker
* envelope thms.
* new: no bordered Hessians

a) Lagrange / KT conditions:
FOC + C / FOC + C + CSC
⇓
candidate pts

b) SOC / EVT (incl. NDCQ) to conclude

a) $L(x,y,z;\lambda) = f(x,y,z) - \lambda(x+y-4z)$

write this out

FOC:
$$\begin{cases} L'_x = f'_x - \lambda \cdot 1 = 0 \\ L'_y = f'_y - \lambda \cdot 1 = 0 \\ L'_z = f'_z + \lambda \cdot 4 = 0 \end{cases}$$

C: $x+y-4z=8$

$$\left(\begin{array}{cccc|c} 10 & -8 & -4 & -1 & 0 \\ -8 & 10 & -4 & -1 & 0 \\ -4 & -4 & 16 & 4 & 0 \\ 1 & 1 & -4 & 0 & 8 \end{array}\right)$$

b) Find candidate pts:

Solve lin. sys. using Gaussian elim.

$$\left(\begin{array}{cccc|c} 1 & 1 & -4 & 0 & 8 \\ 0 & -18 & 36 & 1 & -80 \\ 0 & 0 & 0 & -2 & -16 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

z free

$-2\lambda = -16 \Rightarrow \lambda = 8$

$y = \dfrac{-80-8-36z}{-18}$

$x = -y + 4z + 8$
$= \ldots$

Candidate pts:

$$\left.\begin{array}{l} x = 2z+4 \\ y = 2z+4 \\ z = z \\ \lambda = 8 \end{array}\right\}$$

inf. many candidate pts. for ex: $(4,4,0;8)$ $f = \ldots$

Use SOC:

$(x,y,z;\lambda) = (2z+4, 2z+4, z; 8)$ $\rightsquigarrow$ $L(x,y,z;8) = f(x,y,z) - 8(x+y-4z)$

Check if L is convex:

$H(L) = H(f)$

$\Downarrow$

$L(x,y,z;8)$ is convex since f is convex (from 3a))

$\Longleftarrow$ $(x,y,z) = (2z+4, 2z+4, z)$ is global min

Min value: $f(4,4,0) = \underline{\underline{33}}$

c) Envelope Thm: $\dfrac{df^*(a)}{da} = \dfrac{\partial L}{\partial a}(x^*(a), y^*(a), z^*(a); \lambda^*(a))$

$x+y-4z = a$

$x+y-4z-a=0$

$\longrightarrow$ for $L = f(x,y,z) - \lambda \cdot (x+y-4z-a)$

$$f^*(7.92) \simeq \underset{\substack{\| \\ 33}}{f^*(8)} + \underset{\substack{\| \\ -0.08}}{\Delta a} \cdot \underset{\substack{\| \\ \lambda^*(8)=8}}{\frac{df^*(a)}{da}} = 33 - 0.08 \cdot 8 = \underline{\underline{32.36}}$$

In this solution of 4c) I have assumed that there is a min value $f^*(7.92)$ when $a = 7.92$, and used Envelope Thm to estimate the min value.

In the solutions, I have in many cases showed that there is a min value (this can be a long computation). If you assume there is a min, you could write that to make your assumptions explicit.