LECTURE 2

GRA 6035

MATHEMATICS

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SEP 01, 2017

Plan:

Review Lecture 1
① Matrices and matrix algebra
② Determinants
③ Minors, rank and linear systems

Readings:

[ME] 8.1-8.4, (8.5-8.6), 9.1-9.2, (9.3), 26.1-26.3, (26.4), 26.5

FORK 1003, maths

Exams:

Midterm : Oct 13
Final : Nov 29

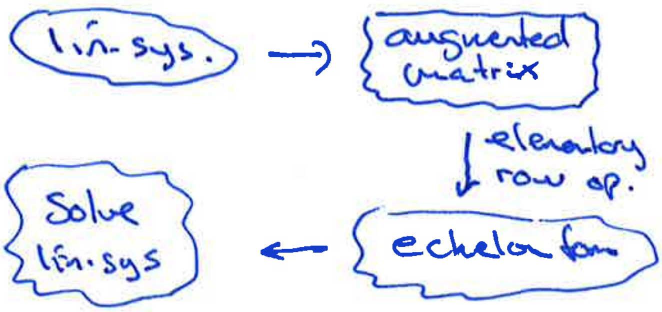
Retake:

Jan 5
(midterm and final)

Exercise sessions: Start next week. ← Tue Sep 05 1700-1900 in C2-020/C2-030

Review: Linear systems and Gaussian elimination
Rank.

- know linear systems
- Gaussian elimination



- know how to find and use pivot positions

* Pivot position determine the solutions of the lin. sys.

* Rank = # pivot positions

① Matrices and matrix algebra.

Defn: An $m \times n$-matrix A is a rectangular array of numbers with m rows, n col's.

$$A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

a_{12}: entry in A in position $(1,2)$

$=$ row 1, col 2

Ex: $A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 4 & 0 \end{pmatrix}$ $\quad a_{23} = 0$

2×3-matrix

Operations:

addition/subtraction: the matrices have the same size, operation position by position

Ex:

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1 & 1 \\ 5 & 7 \end{pmatrix}}}$$

scalar multiplication: scalar = number

position by position

Ex:

$$2 \cdot \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 0 & -2 \\ 4 & 6 \end{pmatrix}}}$$

matrix multiplication: defined if # col(A) = # rows(B)

$$\underset{m\times n}{A} \cdot \underset{n\times p}{B} \longrightarrow \underset{m\times p}{AB}$$

Ex:

$$\underset{2\times3}{\begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & -1 \end{pmatrix}} \cdot \underset{3\times1}{\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix}} = \underline{\underline{\begin{pmatrix} 0 \\ 8 \end{pmatrix}}}$$

$1\cdot2+2\cdot(-1)+3\cdot0$

$4\cdot2+0\cdot(-1)+(-1)\cdot0$

$$\begin{pmatrix} 2 \\ -1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 2 & 3 \\ 4 & 0 & -1 \end{pmatrix}$$ <u>not defined</u>

3×1 — 2×3

Rules for matrices:

$$\boxed{\text{Important: } A\cdot B \neq B\cdot A}$$

i) $A+B=B+A$

ii) $A\cdot(B+C) = A\cdot B + A\cdot C$

iii) $A(B\cdot C) = (A\cdot B)\cdot C$

⋮

Special matrices:

zero matrix: $O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\qquad A+O=A$

identity matrix: $I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$ $\qquad A\cdot I = A$, $\quad I\cdot A = A$

transpose: $A \rightsquigarrow A^T = A^t = A^*$

A: $m \times n$ matrix; A^T: $n \times m$ matrix

interchange rows with columns
||
mirror image along the diagonal

Ex: $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ $\qquad A^T = \begin{pmatrix} 1 & 3 \\ 2 & 4 \end{pmatrix}$

$A = \begin{pmatrix} 1 & -1 & 0 \\ 2 & 2 & 4 \\ 7 & 3 & 0 \end{pmatrix}$ $\qquad A^T = \begin{pmatrix} 1 & 2 & 7 \\ -1 & 2 & 3 \\ 0 & 4 & 0 \end{pmatrix}$

Defn: A is called symmetric if $A^T = A$

$\iff a_{ij} = a_{ji}$ for all $i \neq j$

Ex: $A = \begin{pmatrix} 1 & 7 & -1 \\ 7 & 2 & 3 \\ -1 & 3 & 4 \end{pmatrix}$ is symmetric

Defn: A is square if $\#\text{rows}(A) = \#\text{cols}(A)$

symmetric matrices are square

Rules for transposes:

i) $(A \pm B)^T = A^T \pm B^T$

ii) $(c \cdot A)^T = c \cdot A^T$

iii) $(AB)^T = B^T \cdot A^T$!

Powers: If A is a square matrix, we can define powers A^n of A with $n=1,2,\ldots$

$$A^1 = A$$
$$A^2 = A\cdot A$$
$$A^3 = A\cdot A\cdot A$$
$$\vdots$$

Ex: $A = \begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix}$

$$A^2 = \begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix}\cdot\begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix} = \underline{\begin{pmatrix} -3 & 2 \\ -2 & -4 \end{pmatrix}}$$

$$A^3 = A\cdot A^2 = \begin{pmatrix} 1 & 2 \\ -2 & 0 \end{pmatrix}\cdot\begin{pmatrix} -3 & 2 \\ -2 & -4 \end{pmatrix} = \underline{\begin{pmatrix} -7 & -6 \\ 6 & -4 \end{pmatrix}}$$

$A^{100} = ?$ hard to compute

Inverse matrices:

A square matrix: **Defn:** An inverse matrix of A is a matrix B such that

$$A\cdot B = I$$
$$B\cdot A = I$$

2 has inverse
$2^{-1} = \frac{1}{2}$

Fact: An inverse exists if and only if $|A| \neq 0$, and then the inverse is unique and it is written A^{-1} $(=B)$.

Ex:

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$ad-bc \neq 0$: $A^{-1} = \dfrac{1}{ad-bc} \cdot \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

$ad-bc = 0$: A^{-1} does not exist

In general:

A $n \times n$

$|A| \neq 0 \Rightarrow A^{-1} = \dfrac{1}{|A|} \cdot \operatorname{adj}(A)$, $\operatorname{adj}(A) = C^T$

$|A| = 0$ A^{-1} does not exist (C = cofactor matrix)

Ex:

$$x + y = 4$$
$$-x + 3y = 4$$

$$A \cdot \underline{x} = \underline{b}$$

(matrix form)

$$\begin{array}{ccccc} A & & \underline{x} & & \underline{b} \\ \downarrow & & \downarrow & & \downarrow \\ \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} & \cdot & \begin{pmatrix} x \\ y \end{pmatrix} & = & \begin{pmatrix} 4 \\ 4 \end{pmatrix} \end{array}$$

Using inverse:

$$A\underline{x} = \underline{b}$$
$$A^{-1} \cdot A\underline{x} = A^{-1} \cdot \underline{b}$$
$$I\underline{x} = A^{-1} \cdot \underline{b}$$
$$\underline{x} = A^{-1} \cdot \underline{b}$$

$$|A| = 1 \cdot 3 - 1 \cdot (-1) = 4 \neq 0$$

$$A^{-1} = \frac{1}{4} \cdot \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix}$$

method works when A is square and $|A| \neq 0$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ 4 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 8 \\ 8 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 2 \\ 2 \end{pmatrix}}}$$

② Determinants

$$\underset{\text{n×n-matrix}}{A} \rightsquigarrow \underset{\text{number}}{\det(A) = |A|}$$

$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: $|A| = \underline{ad - bc}$

$$\begin{pmatrix} + & - & + \\ - & + & - \\ + & - & + \end{pmatrix}$$

Methods for computing determinants

i) Cofactor expansion:

Ex: $A = \begin{pmatrix} 1 & 4 & -1 \\ 2 & 0 & 4 \\ 7 & -1 & 0 \end{pmatrix}$

C_{ij}: cofactor in position (i,j)

$\quad \shortparallel$

$(-1)^{i+j} \cdot \boxed{M_{ij}} \leftarrow$ minor

$$|A| = 4 \cdot C_{12} + 0 \cdot C_{22} + (-1) \cdot C_{32}$$

$$= 4 \cdot (-1) \cdot M_{12} + 0 \cdot (+1) \cdot M_{22} + (-1) \cdot (-1) \cdot M_{32}$$

$$= -4 \cdot \begin{vmatrix} 2 & 4 \\ 7 & 0 \end{vmatrix} + 0 \cdot M_{22} - (-1) \cdot \begin{vmatrix} 1 & -1 \\ 2 & 4 \end{vmatrix}$$

$$= -4 \cdot (-28) + 0 + 1 \cdot (6) = 112 + 6 = \underline{\underline{118}}$$

General method – (can be used to compute n×n det. for any n)

– Same answer along any row/col.

Defn: M_{ij} = minor obtained by deleting row i, col j

= determinant of the submatrix obtained by deleting row i, col j.

Ex:

$$A = \begin{pmatrix} \cancel{1} & \cancel{\textcircled{4}} & \cancel{1} \\ 2 & \cancel{0} & 4 \\ 7 & \cancel{-1} & 0 \end{pmatrix} \qquad M_{12} = \begin{vmatrix} 2 & 4 \\ 7 & 0 \end{vmatrix}$$

$$= 2 \cdot 0 - 4 \cdot 7 = \underline{-28}$$

ii) Gaussian elimination: simplify the determinant using elementary row op.

Ex:

$$A = \begin{pmatrix} \textcircled{1} & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{pmatrix} \begin{matrix} \rfloor -1 & \\ & \rfloor -1 \end{matrix} \rightarrow \begin{pmatrix} \textcircled{1} & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} \cancel{1} & \cancel{1} & \cancel{1} \\ \cancel{0} & -2 & 0 \\ \cancel{0} & 1 & 3 \end{vmatrix} = +1 \cdot (-2 \cdot 3) = \underline{\underline{-6}}$$

Fact: A → B elementary row operation

i) If you add a multiple of one row to another row, then $|B| = |A|$.

ii) If you multiply a row by c, then $|B| = c \cdot |A|$

iii) If you interchange two rows, then $|B| = -|A|$

Properties of the determinant:

i) $|AB| = |A| \cdot |B|$

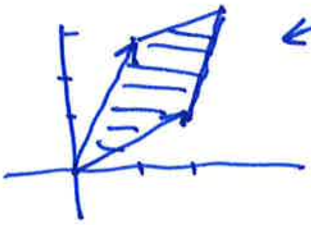
ii) $|c \cdot A| = c^n \cdot |A|$ (if A is nxn)

iii) $|A^T| = |A|$

iv) $|A^{-1}| = 1/|A|$

Ex:

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 3 \end{pmatrix} \qquad |A| = 5$$



area of the parallellogram

If A is a diagonal matrix or a upper/lower triangular matrix, then the determinant is the product of the diagonal entries:

$$|A| = a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$$

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 7 \end{pmatrix}$$

$$A = \begin{pmatrix} \textcircled{1} & 4 & -1 \\ 0 & \textcircled{2} & 3 \\ 0 & 0 & \textcircled{7} \end{pmatrix}$$

Ex:

$$|A| = 1 \cdot 2 \cdot 7$$
$$= \underline{14}$$

Ex: Compute

$$\begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 1 & -1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{vmatrix}.$$

Hint: Use some elementary row operations to simplify first.

Sol:

$$\begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 1 & -1 & 2 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \end{vmatrix} \begin{matrix} \\ -\frac{1}{4} \\ \\ -1/4 \\ \\ \end{matrix}$$

$$= \begin{vmatrix} 4 & 0 & 0 & -1 & -1 \\ 0 & 2 & 0 & 1 & -1 \\ 0 & 0 & 6 & -2 & 0 \\ 0 & -1 & 2 & 1/4 & 1/4 \\ 0 & 1 & 0 & 1/4 & 1/4 \end{vmatrix}$$

$$= 4 \begin{vmatrix} 2 & 0 & 1 & -1 \\ 0 & 6 & -2 & 0 \\ -1 & 2 & 1/4 & 1/4 \\ \textcircled{1} & 0 & 1/4 & 1/4 \end{vmatrix} \begin{matrix} \\ -2 \\ \\ 1 \end{matrix}$$

$$= 4 \cdot \begin{vmatrix} 0 & 0 & 1/2 & -3/2 \\ 0 & 6 & -2 & 0 \\ 0 & 2 & 1/2 & 1/2 \\ \textcircled{1} & 0 & 1/4 & 1/4 \end{vmatrix}$$

$$= 4 \cdot 1 \cdot (-1) \cdot \begin{vmatrix} 0 & 1/2 & -3/2 \\ 6 & -2 & 0 \\ 2 & 1/2 & 1/2 \end{vmatrix}$$

$$= -4 \cdot \left(-6 \cdot \left(1/4 + 3/4 \right) + 2(-3) \right)$$

$$= -4 \cdot (-6 - 6) = -4 \cdot (-12) = \underline{\underline{48}}$$

③ Minors and rank

A mxn matrix $\longrightarrow$ **Defn:** An r-minor or a minor of order r is the determinant of an r×r-submatrix of A.

Ex: $A = \begin{pmatrix} 1 & 2 & 4 \\ 7 & -1 & 3 \end{pmatrix}$ 2x3

Maximal minor: 2-minors

$M_{12,12} = \begin{vmatrix} 1 & 2 \\ 7 & -1 \end{vmatrix} = \underline{-15} \neq 0$

$M_{12,23} = \begin{vmatrix} 2 & 4 \\ -1 & 3 \end{vmatrix} = \underline{10}$

$M_{12,13} = \begin{vmatrix} 1 & 4 \\ 7 & 3 \end{vmatrix} = \underline{-25}$

$\underline{\underline{\text{rank}(A) = 2}}$

1-minors: Coeff. from the matrix

$\underline{1} \quad \underline{2} \quad \underline{4}$

$\underline{7} \quad \underline{-1} \quad \underline{3}$

Rank: The rank of a matrix is the maximal order of a non-zero minor.

Special case:

A nxn matrix $\longrightarrow$ $|A|$ maximal minor

$|A| \neq 0 : rk(A) = n$

$|A| = 0 : rk(A) \leq n-1$

Ex: $A = \begin{pmatrix} \textcircled{1} & 1 & 1 \\ 1 & \textcircled{-1} & 1 \\ 1 & 2 & \textcircled{4} \end{pmatrix}$

$|A| = 1\cdot(-6) - 1\cdot 3 + 1\cdot 3 = \underline{-6 \neq 0}$

$\Rightarrow \text{rk } A = \underline{\underline{3}}$

pivot positions

⇓

one solution of the linear system

$$\begin{aligned} \underline{x} + y + z &= 3 \\ x - \underline{y} + z &= 1 \\ x + 2y + \underline{4z} &= 7 \end{aligned}$$

Note: This applies regardless of the last column

$$\begin{pmatrix} 3 \\ 1 \\ 7 \end{pmatrix}$$

since there cannot be a pivot in this column.

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 3 & 5 & -1 \end{pmatrix}$

3-minors = maximal minors

$|A| = 1\cdot(-2+5) - 1(-1+3) + 1\cdot(5-6)$

$= 3 - 2 - 1 = \underline{0} \quad \rightarrow \underline{\text{rk } A \neq 3}$

2-minors:

random choice of 2-minor → $M_{12,12} = \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 2 - 1 = 1 \neq 0$

$\rightarrow \underline{\text{rk}(A) = 2}$

pivot positions in A:

$$\begin{pmatrix} \textcircled{1} & 1 & 1 \\ 1 & \textcircled{2} & -1 \\ 3 & 5 & -1 \end{pmatrix}$$

Ex: Linear system with coeff. matrix $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & -1 \\ 3 & 5 & -1 \end{pmatrix}$:

$$\left.\begin{aligned} x + y + z &= 7 \\ x + 2y - z &= 5 \\ 3x + 5y - z &= 17 \end{aligned}\right\}$$

rk $A = 2$ with pivot pos. at $\begin{pmatrix} \textcircled{-} & - & \cdot \\ - & \textcircled{-} & \cdot \\ - & - & \cdot \end{pmatrix}$

$\Downarrow$

either i) inf. many solutions with z free variable, or

ii) no solutions.

We forgot to check this in the lecture. →

It depends on $\underline{b} = \begin{pmatrix} 7 \\ 5 \\ 17 \end{pmatrix}$, last column; Must check if there is pivot in last col, i.e. if rk A = rk $(A|\underline{b})$.

$$(A|\underline{b}) = \begin{pmatrix} 1 & 1 & 1 & 7 \\ 1 & 2 & -1 & 5 \\ 3 & 5 & -1 & 17 \end{pmatrix}$$

3-minors:

$M_{123,123} = |A| = 0$

$$M_{123,124} = \begin{vmatrix} 1 & 1 & 7 \\ 1 & 2 & 5 \\ 3 & 5 & 17 \end{vmatrix} = ~~[illegible]~~ \quad 9 - 2 - 7 = \underline{0}$$

$$M_{123,134} = \begin{vmatrix} 1 & 1 & 7 \\ 1 & -1 & 5 \\ 3 & -1 & 17 \end{vmatrix} = -12 - 2 + 14 = \underline{0}$$

$$M_{123,234} = \begin{vmatrix} 1 & 1 & 7 \\ 2 & -1 & 5 \\ 5 & -1 & 17 \end{vmatrix} = -12 - 9 + 21 = \underline{0}$$

three new 3-minors to compute

$$\Rightarrow \text{rk}\,(A|\underline{b}) = \text{rk}\,(A) = \underline{2}$$

Concl: ok, there are inf. many sol. with free variables $\textcircled{z}$

$$\boxed{x+y}+z = 7$$
$$\boxed{x+2y}-z = 5$$
$$\cancel{3x+5y-z = 17}$$

From our computations, we know that all 3-minors in A, (A|b) are zero, and the 2-minor is non-zero: $\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \neq 0$

$\Downarrow$

Last equation is irrelevant, x, y basic, z free.

Solution:

$$x+y = 7-z$$
$$x+2y = 5+z$$

$$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7-z \\ 5+z \end{pmatrix}$$

know that the inverse exists since $\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} = 1 \neq 0$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 7-z \\ 5+z \end{pmatrix}$$

$$= \frac{1}{1} \cdot \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 7-z \\ 5+z \end{pmatrix} = \begin{pmatrix} 2(7-z) - 1\cdot(5+z) \\ -1(7-z) + 1(5+z) \end{pmatrix}$$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 9-3z \\ -2+2z \end{pmatrix}$$

Explicit solution:

$x = 9-3z$
$y = -2+2z$
$z = z$ (free)

Note:

If the last equation had been

$$3x+5y-z = 17$$

had constant ~~term~~ $\neq 17$, then it would have no solution.

Ex: $x+y+z=7$
$x+2y-z=5$
$3x+5y-z=1$

no solutions since

$\text{rk} A = 2 < \text{rk}(A|b) = 3$

$(M_{123,124} = -16 \neq 0)$