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LECTURE 6

GRA 6035 MATHEMATICS

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SEP 29, 2017

Plan:

1. Definiteness of symmetric matrices
2. Unconstrained optimization in n variables
3. Convex and concave functions

Reading:

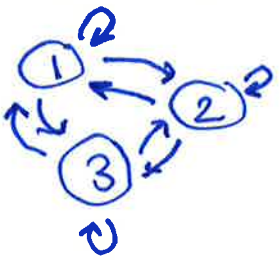
[MEJ 14.1-14.4, 14.8, 17.1-17.5]

Review:

Markov chains

$\underline{x}$: state vector
col. sum = 1

$$\boxed{A \cdot \underline{x}_t = \underline{x}_{t+1}}$$



$\rightsquigarrow$ $A = (a_{ij})$
transition matrix

a_{ij}: $(j) \to (i)$
col. sums = 1

Equilibrium: $\lim\limits_{t\to\infty} \underline{x}_t = \lim\limits_{t\to\infty} A^t \cdot \underline{x}_0$

Thm:
If A is regular, then
* there is a unique eigenvector $\underline{u}$ with $\lambda = 1$ s.t. $\underline{u}$ is a state vector
* $\underline{u}$ is the equilibrium state

Defn: A regular if
A^n has non-zero entries for n big enough

$$\boxed{A \text{ non-zero entries} \Rightarrow A \text{ regular}}$$

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Quadratic forms:

$f(\underline{x}) = f(x_1,\dots,x_n) = \underline{x}^T \cdot A \cdot \underline{x}$ for a unique symmetric $n\times n$-matrix A

polynomial with all terms of degree 2.

Ex: $f(x,y,z) = x^2 + 6xz - y^2 = (x\ y\ z) \cdot \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 0 \\ 3 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

$\underline{x}^T$ A $\underline{x}$

the symm. matrix of f.

$g(x,y,z) = 2x^2 + y^2 + 3z^2 = \underline{x}^T \cdot \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \cdot \underline{x}$

Defn: A symm matrix of f and $\lambda_1,\dots,\lambda_n$ are eigenvalues of A

f pos. semidefn. $\iff f(\underline{x}) \geq 0$ for all $\underline{x}$ $\iff \lambda_1,\lambda_2,\dots,\lambda_n \geq 0$

f pos. defn. $\iff f(\underline{x}) > 0$ for all $\underline{x} \neq \underline{0}$ $\iff \lambda_1,\lambda_2,\dots,\lambda_n > 0$

f neg. semidefn. $\iff f(\underline{x}) \leq 0$ for all $\underline{x}$ $\iff \lambda_1,\lambda_2,\dots,\lambda_n \leq 0$

f neg. defn. $\iff f(\underline{x}) < 0$ for all $\underline{x} \neq \underline{0}$ $\iff \lambda_1,\lambda_2,\dots,\lambda_n < 0$

f indefinite $\iff f(\underline{x})$ has pos. and neg. values $\iff$ we have both pos. and neg. eigenvalues

defn.   result

Ex: $g(x,y,z) = 2x^2 + y^2 + 3z^2 \geq 0$ $\quad A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ $\quad \lambda_1 = 2 \geq 0,\ \lambda_2 = 1 \geq 0,\ \lambda_3 = 3 \geq 0$ $\quad$ pos. semidefinite (pos. definite also)

$h(x,y,z) = 2x^2 - y^2 + z^2$ $\quad \lambda_1 = 2,\ \lambda_2 = -1,\ \lambda_3 = 1$ $\quad$ indefinite

$h(1,0,0) = 2$

$h(0,1,0) = -1$

Result is based on the fact that any quadratic form can be re-written as

$$f(\underline{x}) = \lambda_1 \cdot u_1^2 + \lambda_2 \cdot u_2^2 + \dots + \lambda_n \cdot u_n^2$$

where $u_1,\dots,u_n$ is linear expressions in $x_1,\dots,x_n$.

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(1) Definiteness of symmetric matrices : A symmetric $n \times n$ matrix

Method: Principal minors

(a) Leading principal minors:

$D_1 = M_{1,1}$

$D_2 = M_{12,12}$

$D_3 = M_{123,123}$

$\vdots$

$D_n = M_{12\ldots n, 12\ldots n} = |A|$

$$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -1 & 0 \\ 3 & 0 & 0 \end{pmatrix} \qquad D_1 = 1, \quad D_2 = -1, \quad D_3 = 3\cdot 3 = 9$$

$f(x,y,z) = x^2 + 6xz - y^2$

$= \underset{1}{(x+3z)^2} \underset{-1}{- y^2} \underset{-9}{- 9z^2} \Rightarrow$ indefinite

Result:

A pos. definite $\overset{\text{defn.}}{\Longleftrightarrow}$ $\underline{x}^T A \cdot \underline{x} > 0$ for $\underline{x} \neq \underline{0}$ $\Longleftrightarrow$ $D_1, D_2, \ldots, D_n > 0$

A neg. definite $\Longleftrightarrow$ $\underline{x}^T A \cdot \underline{x} < 0$ — " — $\Longleftrightarrow$ $D_1 < 0, D_2 > 0, D_3 < 0, \ldots$

Ex: $f(\underline{x}) = -x^2 - y^2 - z^2$

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$D_1 = -1 \leftarrow \lambda_1$

$D_2 = 1 \leftarrow \lambda_1 \cdot \lambda_2$

$D_3 = -1 \leftarrow \lambda_1 \cdot \lambda_2 \cdot \lambda_3$

$\lambda_1 = -1 \quad \lambda_2 = -1 \quad \lambda_3 = -1 < 0$

neg. definite

What if A is neither pos. or neg. definite?

fails:
- wrong sign $\Rightarrow$ indefinite
- zero $\Rightarrow$ indefinite or pos/neg. semidefinite.

In general, we try to determine the definiteness using leading principal minors $D_1, \ldots, D_n$. If this is not possible, we look at

All principal minors

Δ_i : minor of order i obtained by choosing i rows, and the same i columns. = deleting all other rows and columns.

Result:

A pos. semidefn. $\iff$ $\Delta_1, \Delta_2, \ldots, \Delta_n \geq 0$

A neg. semidefn. $\iff$ $\Delta_1 \leq 0$, $\Delta_2 \geq 0$, $\Delta_3 \leq 0, \ldots$

A indefinite $\iff$ all other cases

Typical examples:

$D_2 < 0$ or $\Delta_2 < 0$: indefinite

Δ_1 and Δ_3 opposite signs : indefinite

Ex:

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

$D_1 = 1$
$D_2 = 0$
$D_3 = 1 \cdot 0 - 2 \cdot 0 = 0$

either pos. semid. or indefinite

$\Delta_1 = 1, 4, 0 \geq 0$
$\Delta_2 = 0, -4, -1$ $\Rightarrow$ indefinite
$\Delta_3 = 0$

D_i

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The case of $|A|=0$:

Thm: (New result, 2016)

If $\operatorname{rk} A = r < n$, ~~then~~ we have:

$D_1>0, D_2>0, \dots, D_r>0$: A pos. semidefinite

$D_1<0, D_2>0, \dots, (-1)^r D_r>0$: A neg. semidefinite

Ex:

$$A = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \qquad \begin{aligned} D_1 &= 1 \\ D_2 &= 1 \\ D_3 &= 0 \\ D_4 &= 0 \end{aligned} \Rightarrow A \text{ pos. semidefin.}$$

(using new result)

What is $\operatorname{rk} A$? $\operatorname{rk} A = 2$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} \begin{matrix} \\ \\ -1 \\ -1 \end{matrix} \rightarrow \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\operatorname{rk} A = 2 \Rightarrow$ All minors of order 3 and 4 are zero!

Alt: Using principal minors: "old method"

$\Delta_1 = 1, 1, 1, 1 \geq 0$

$\Delta_2 = 0, 1, 0, 0, 1, 1 \geq 0$

$\Delta_3 = 0, 0, 0, 0 \geq 0$

$\Delta_4 = 0 \geq 0$

pos. semidefinite

(6 2-minors, 4 3-minors, 1 4-minor = determinant to compute!)

2 Unconstrained optimization in n vars.

max/min $f(x_1, \ldots, x_n) = f(\underline{x})$ (no constraints)

$\boxed{\begin{array}{l} \text{max} = \text{global max} \\ \text{min} = \text{global min} \end{array}}$

Method:

1. Find all stationary pts: $\boxed{f'_{x_1} = f'_{x_2} = \ldots = f'_{x_n} = 0}$ FOC = first order cond.

2. Compute the Hessian matrix

$$H(f) = \begin{pmatrix} f''_{x_1x_1} & f''_{x_1x_2} & \cdots & f''_{x_1x_n} \\ \vdots & \vdots & \ddots & \vdots \\ f''_{x_nx_1} & f''_{x_nx_2} & \cdots & f''_{x_nx_n} \end{pmatrix} \leftarrow n \times n \text{ symmetric matrix}$$

true if f is "nice", i.e. C^2 = all second order partial derivatives exist and are cont.

and classify the stationary pts $\underline{x}^*$:

Second derivative test:

- $H(f)(\underline{x}^*)$ pos. defin. $\Rightarrow \underline{x}^*$ is local min
- $H(f)(\underline{x}^*)$ neg. defin. $\Rightarrow \underline{x}^*$ is local max
- $H(f)(\underline{x}^*)$ indefinite $\Rightarrow \underline{x}^*$ is saddle pt

inconclusive if pos. semidefin. but not pos. defin.
— " — neg. — " — neg — " —

Ex. $f(x,y) = x^3 - 3xy + y^3$

$f'_x = 3x^2 - 3y = 0$

$f'_y = -3x + 3y^2 = 0$

$\cancel{3}x^2 = \cancel{3}y \qquad \underline{y = x^2}$

$-3x + 3\cdot(x^2)^2 = 0$

$-3x + 3x^4 = 0$

$-3x(1 - x^3) = 0$

$\underline{x=0}$ or $x^3 = 1$, $\underline{x=1}$

$x=0: y=0 \to \underline{(0,0)}$

$x=1: y=1 \to \underline{(1,1)}$

Stat. points: $(0,0), (1,1)$

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$f(x,y) = x^3 - 3xy + y^3$ Stat. pt: $(0,0), (1,1)$

$f'_x = 3x^2 - 3y$

$f'_y = -3x + 3y^2$

$$\left.\begin{matrix} f''_{xx} = 6x & f''_{xy} = -3 \\ f''_{yx} = -3 & f''_{yy} = 6y \end{matrix}\right\} \quad H(f) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$$

Classification as local max, local min or saddle pts

Defn: $\underline{x}^* = (x_1^*, x_2^*, \ldots, x_n^*)$ stationary pt:

$\underline{x}^*$ local max: $f(\underline{x}^*) \geq f(\underline{x})$ for all $\underline{x}$ near $\underline{x}^*$

$\underline{x}^*$ local min: $f(\underline{x}^*) \leq f(\underline{x})$ for all $\underline{x}$ near $\underline{x}^*$

$\underline{x}^*$ saddle pt: otherwise

$(x,y) = (1,1)$: $H(f)(1,1) = \begin{pmatrix} 6 & -3 \\ -3 & 6 \end{pmatrix} = \begin{pmatrix} A & B \\ B & C \end{pmatrix}$

$D_1 = 6 > 0$ A

$D_2 = 36 - 9 = 27 > 0 \longleftarrow AC - B^2$

$\Downarrow$

$(1,1)$ is local min. $f(1,1) = -1$

$H(f)(1,1)$ pos. defn.

$(x,y) = (0,0)$: $H(f)(0,0) = \begin{pmatrix} 0 & -3 \\ -3 & 0 \end{pmatrix}$

$\left.\begin{matrix} D_1 = 0 \\ D_2 = -9 \end{matrix}\right\}$ $H(f)(0,0)$ indefinite

$\Downarrow$

$(0,0)$ is saddle pt.

Max/min = global max/min:

global max/min $\Rightarrow$ local max/min $\Rightarrow$ stationary pt.

Ex: $f(x,y) = x^3 - 3xy + y^3$

$(x,y) = (1,1)$ may be global min

no global max.

In fact, no global min either.

$f(-2,0) = -8 < -1$

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3 Convex and concave functions

f convex : any local min is a global min
any stationary pt. is a global min

f concave : any stationary pt is a global max

Defn:

f is convex if  $\iff$ $H(f)(\underline{x})$ is pos. semidefinite for all $\underline{x}$.

f is concave if  $\iff$ $H(f)(\underline{x})$ is neg. semidefinite for all $\underline{x}$.

Ex: $H(f) = \begin{pmatrix} 6x & -3 \\ -3 & 6y \end{pmatrix}$ $\quad D_1 = 6x$, $D_2 = 36xy - 9$

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General defn. in full details:

f is convex if:

For any two points on the graph of f, the line segment joining these pts are on or over the graph of f.

f is concave if:

For any two points on the graph of f, the line segment joining these pts are on or under the graph of f.

These defn. can be used in all cases. When f is a nice function with derivatives (f is C^2), then we use $H(f)$ to test if f is convex/concave.

Ex: $f(x,y) = e^{2x-3y}$

$u = 2x-3y$

$$\left.\begin{array}{l} f'_x = 2 \cdot e^{2x-3y} \\ f'_y = -3 \cdot e^{2x-3y} \end{array}\right\} \Rightarrow H(f) = \begin{pmatrix} 4e^{2x-3y} & -6e^{2x-3y} \\ -6e^{2x-3y} & 9e^{2x-3y} \end{pmatrix} = \begin{pmatrix} 4e^u & -6e^u \\ -6e^u & 9e^u \end{pmatrix}$$

$\Delta_1 = 4e^u, 9e^u > 0$

$\Delta_2 = 0$

$$\Leftarrow \begin{cases} D_1 = 4e^u > 0 \\ D_2 = 36e^{2u} - 36e^{2u} = 0 \geq 0 \end{cases}$$

pos. semidefn. for all $\underline{x}$ $\Rightarrow$ f is convex

An Example

Let $f(x,y) = x^2y^3 + y^2 - 2y$. Then f has just one stationary pt, which is a local min. However, f has no global min.

$$\left.\begin{aligned} f'_x &= 2xy^3 = 0 \\ f'_y &= 3x^2y^2 + 2y - 2 = 0 \end{aligned}\right\}$$

FOC: $x = 0$ or $y = 0$

If $x = 0$, then (2) gives

$2y - 2 = 0 \Rightarrow y = 1$

If $y = 0$, then (2) gives

$-2 = 0$ no solution

$$H(f) = \begin{pmatrix} 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y + 2 \end{pmatrix}$$

$$H(f)(0,1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} \quad \begin{aligned} D_1 &= 2 > 0 \\ D_2 &= 4 > 0 \end{aligned}$$

pos. defn. $\Rightarrow$ $(0,1)$ local min

Conclusion: $(0,1)$ is the only stationary pt, with $f(0,1) = -1$

But $(x,y) = (0,1)$ with $f(0,1) = -1$ is not global min

For example,

$$f(3,-1) = -9 + 1 + 2 = -6 < -1$$

Conclusion: f has no global min even if $(0,1)$ is local min and this is the only stationary point.