LECTURE 8

GRA 6035 BI

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OCT 13, 2017

MATHEMATICS

Plan:

Review Lecture 7

(1) Second order conditions

(2) Non-degenerate constraint qualification

Reading:

[MEJ] 18.1-18.7, (12.3-12.5), 21.1, 19.1, 19.4

Review: Constrained optimization

Lagrange problems	Kuhn-Tucker problems
max/min $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$, where $f(\underline{x}) = f(x_1, \dots, x_n)$	max $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) \le a_1 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{cases}$ (std. form)
	(min and/or $g(\underline{x}) \ge a$ $\rightarrow$ rewrite to std. form)
$\mathcal{L} = f(\underline{x}) - \lambda_1 \cdot g_1(\underline{x}) - \dots - \lambda_m \cdot g_m(\underline{x})$	
FOC: $\boxed{\mathcal{L}'_{x_1} = 0 \;\; \vdots \;\; \mathcal{L}'_{x_n} = 0}$	FOC: $\boxed{\mathcal{L}'_{x_1} = 0 \;\; \vdots \;\; \mathcal{L}'_{x_n} = 0}$
C: $\boxed{g_1(\underline{x}) = a_1 \;\; \vdots \;\; g_m(\underline{x}) = a_m}$	C: $\boxed{g_1(\underline{x}) \le a_1 \;\; \vdots \;\; g_m(\underline{x}) \le a_m}$
	CSC: $\boxed{\lambda_1 \ge 0 \text{ and } \lambda_1 \cdot (g_1(\underline{x}) - a_1) = 0 \;\; \vdots \;\; \lambda_m \ge 0 \text{ and } \lambda_m (g_m(\underline{x}) - a_m) = 0}$
Lagrange conditions: FOC + C	Kuhn-Tucker conditions: FOC + C + CSC
Solve FOC + C : $(x_1^*, x_2^*, \dots, x_n^*; \lambda_1^*, \lambda_2^*, \dots, \lambda_m^*)$ $\Downarrow$ Candidates for max/min	Solve FOC + C + CSC $\Downarrow$ Candidates for max

Theorem (Necessary conditions)

If $\underline{x}^* = (x_1^*, x_2^*, \dots, x_n^*)$ is a solution of the Lagrange (Kuhn-Tucker) problem, then there are Lagrange multipliers $\lambda_1, \dots, \lambda_m$ such that

$$(x_1^*, \dots, x_n^*; \lambda_1, \dots, \lambda_m)$$

satisfies FOC+C (FOC+C+CSC), assuming that NDCQ (non-degenerate constraint qualification) holds.

$$\underline{x}^* \text{ is max/min} \underset{\text{NDCQ}}{\Longrightarrow} (x^*; \lambda^*) \text{ satisfies FOC+C / FOC+C+CSC.}$$

Ex: max $f = x + 3y$ when $x^2 + y^2 \leq 10$ (KT, std. form)

$\mathcal{L} = f - \lambda g = x + 3y - \lambda(x^2 + y^2)$

FOC: $\mathcal{L}'_x = 1 - \lambda \cdot 2x = 0$

$\mathcal{L}'_y = 3 - \lambda \cdot 2y = 0$

C: $x^2 + y^2 \leq 10$: a) $x^2+y^2 = 10$, b) $x^2 + y^2 < 10$

	a)	b)
C:	$x^2 + y^2 = 10$	$x^2 + y^2 < 10$
CSC:	$\lambda \geq 0$	$\lambda = 0$

a) $x = \frac{1}{2\lambda}$

$y = \frac{3}{2\lambda}$

$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 = 10$

$\frac{10}{(2\lambda)^2} = \frac{1+9}{(2\lambda)^2} = 10$

$2\lambda = \pm 1$

$\lambda = \pm 1/2 = 1/2$

$x = 1$

$y = 3$

$\Downarrow$

$(x, y; \lambda) = (1, 3; 1/2)$

candidate pt.

b) $\lambda = 0$: $1 = 0$

$\Rightarrow$ no solutions

Conclusion so far:

$(x,y;\lambda) = (1,3;1/2)$ is the only candidate pt. for $\max x+3y$ when $x^2+y^2 \le 10$

$f(1,3)=10$

① Second order conditions:

$(x,y;\lambda)=(1,3;1/2)$: $\quad L(x,y;1/2) = x+3y - \frac{1}{2}(x^2+y^2)$

check if $L(x,y;1/2)$ is concave:

$$L'_x = 1 - \tfrac{1}{2}\cdot 2x = 1-x$$
$$L'_y = 3 - \tfrac{1}{2}\cdot 2y = 3-y$$

$$H(L) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$D_1 = -1 < 0$

$D_2 = 1 > 0$

$L(x,y;1/2)$ is concave $\Longleftarrow$ $H(L)(x,y)$ is negative definite for all (x,y)

SOC (second order condition) $\Rightarrow$ $(x,y)=(1,3)$ is max

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Theorem: (SOC = second order condition)

If $(x_1^*,\dots,x_n^*;\lambda_1^*,\dots,\lambda_m^*)$ satisfies FOC+C / FOC+C+CSC in a Lagrange / Kuhn–Tucker problem, then the following holds:

i) If $L(x_1,\dots,x_n;\lambda_1^*,\dots,\lambda_m^*)$ is concave, then $(x_1^*,\dots,x_n^*)$ is a maximum pt.

ii) If $L(x_1,\dots,x_n;\lambda_1^*,\dots,\lambda_m^*)$ is convex, then $(x_1^*,\dots,x_n^*)$ is a minimum pt.

Ex: $\min\ 2x^2+y^2+3z^2$ when $\begin{array}{ll} x-y+2z \geq 3 \\ x+y \geq 3 \end{array}$

$\Updownarrow$

$\max\ -(2x^2+y^2+3z^2)$ when $\begin{cases} -(x-y+2z) \leq -3 \\ -(x+y) \leq -3 \end{cases}$ KT, Std. form

$$\mathcal{L} = -(2x^2+y^2+3z^2) + \lambda_1(x-y+2z) + \lambda_2(x+y)$$

FOC: $\boxed{\begin{cases} \mathcal{L}'_x = -4x + \lambda_1 + \lambda_2 = 0 \\ \mathcal{L}'_y = -2y - \lambda_1 + \lambda_2 = 0 \\ \mathcal{L}'_z = -6z + 2\lambda_1 = 0 \end{cases}}$

C:	a) $x-y+2z=3$, $x+y=3$	b) $x-y+2z=3$, $x+y>3$	c) $x-y+2z>3$, $x+y=3$	d) $x-y+2z>3$, $x+y>3$
CSC:	$\lambda_1 \geq 0$ $\lambda_2 \geq 0$ (see next page) $x=2$ $y=1$ $z=1$ $\lambda_1=3$ $\lambda_2=5$ $(x,y,z;\lambda_1,\lambda_2)$ $=(2,1,1;3,5)$	$\lambda_1 \geq 0$ $\lambda_2 = 0$	$\lambda_1 = 0$ $\lambda_2 \geq 0$	$\lambda_1 = 0$ $\lambda_2 = 0$ $\Downarrow$ $x=0$ $y=0$ $z=0$ $x-y+2z=0 \not> 3$ $\Downarrow$ no sol's in this case.

a) $\left.\begin{aligned} -4x + \lambda_1 + \lambda_2 &= 0 \\ -2y - \lambda_1 + \lambda_2 &= 0 \\ -6z + 2\lambda_1 &= 0 \end{aligned}\right\}$ FOC

$\left.\begin{aligned} x - y + 2z &= 3 \\ x + y &= 3 \end{aligned}\right\}$ C

$\left.\begin{aligned} \lambda_1 &\geq 0 \\ \lambda_2 &\geq 0 \end{aligned}\right\}$ CSC

$x = \frac{1}{4}\cdot 3 + \frac{1}{4} 5 = \underline{2}$

$y = -\frac{1}{2}\cdot 3 + \frac{1}{2}\cdot 5 = \underline{1}$

$z = \frac{1}{3} 3 = \underline{1}$

$\underline{\lambda_1 = 3}$

$\underline{\lambda_2 = 5}$

<u>FOC+C:</u>

$$\left(\begin{array}{ccccc|c} -4 & 0 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & -6 & 2 & 0 & 0 \\ 1 & -1 & 2 & 0 & 0 & 3 \\ 1 & 1 & 0 & 0 & 0 & 3 \end{array}\right) \begin{array}{l} :(-4) \\ :(-2) \\ :(-6) \\ \\ \\ \end{array}$$

$$\to \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -1/4 & -1/4 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & -1/3 & 0 & 0 \\ 1 & -1 & 2 & 0 & 0 & 3 \\ 1 & 1 & 0 & 0 & 0 & 3 \end{array}\right) \begin{array}{l} -1 \\ -1 \end{array}$$

$$\to \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -1/4 & -1/4 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & -1/3 & 0 & 0 \\ 0 & -1 & 2 & 1/4 & 1/4 & 3 \\ 0 & 1 & 0 & 1/4 & 1/4 & 3 \end{array}\right) \begin{array}{l} 1 \\ -1 \end{array}$$

$$\begin{array}{l} -2 \\ -1 \end{array} \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -1/4 & -1/4 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & -1/3 & 0 & 0 \\ 0 & 0 & 2 & 3/4 & -1/4 & 3 \\ 0 & 0 & 0 & -1/4 & 3/4 & 3 \end{array}\right)$$

$$\to \left(\begin{array}{ccccc|c} 1 & 0 & 0 & -1/4 & -1/4 & 0 \\ 0 & 1 & 0 & 1/2 & -1/2 & 0 \\ 0 & 0 & 1 & -1/3 & 0 & 0 \\ 0 & 0 & 0 & 17/12 & -1/4 & 3 \\ 0 & 0 & 0 & -1/4 & 3/4 & 3 \end{array}\right)$$

$\frac{17}{12}\lambda_1 - \frac{1}{4}\lambda_2 = 3$ $\;\; 3$

$-\frac{1}{4}\lambda_1 + \frac{3}{4}\lambda_2 = 3$

$\cdot 4$

$4\cdot\frac{17}{\cancel{12}}\cdot\cancel{3}\lambda_1 - \frac{1}{4}\lambda_1 = 12$

$17\lambda_1 - \lambda_1 = 48$

$16\lambda_1 = 48 \quad \lambda_1 = 3$

$-\lambda_1 + 3\lambda_2 = 12 \quad 3\lambda_2 = 15 \quad \underline{\lambda_2 = 5}$

max $-(2x^2+y^2+3z^2)$ when $x-y+2z \geq 3$, $x+y \geq 3$ $\Rightarrow$ $-(x-y+2z) \leq -3$, $-(x+y) \leq -3$

BI

<u>Candidate:</u> $(2,1,1;3,5)$

<u>SOC:</u> $\mathcal{L}(x,y,z;3,5) = -2x^2-y^2-3z^2+3(x-y+2z)+5(x+y)$

is this concave in (x,y,z)?

$$H(\mathcal{L}) = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

$D_1 = -4$
$D_2 = 8$
$D_3 = -48$

negative defn $\Downarrow$ $\mathcal{L}(x,y,z;3,5)$ is <u>concave</u>

SOC

$(x,y,z) = (2,1,1)$ is max in KT (std. fm)

<u>Max value:</u> $-(8+1+3) = -12$

<u>Conclusion:</u> $(x,y,z) = (2,1,1)$ is the minimum pt, with minimum value $\underline{\underline{f = 12}}$

② NDCQ and extreme value theorem

Extreme value thm:

If f is cont. function defined on a closed and bounded set S, then f has a max and min on S.

L: max/min $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$

f is cont.

$S = \{\underline{x} : g_1(\underline{x}) = a_1, \ldots, g_m(\underline{x}) = a_m\}$ is closed set of adm. points

Must check that S is bounded

If so, there is a max/min

$\Downarrow$

$\underline{x}^*$ max $\underset{\text{NDCQ}}{\Longrightarrow}$ $(\underline{x}^*, \lambda^*)$ satisfy FOC+C

Two possibilities:

i) The max/min is one of the candidate pts.

ii) The max/min is an adm. pt where NDCQ fails.

KT: Similar.

Method for solving Lagrange / Kuhn-Tucker problems (alternative to SOC)

① Check if S = set of adm. pts is bounded.

② Find all candidate pts.
Find all adm. pts where NDCQ fails (if any)
Compute and compare function values.

NDCQ: Non-degenerate constraint qualification

Ex: $\underbrace{x^2+y^2}_{g(x,y)} = \underbrace{10}_{a}$

$$J = \begin{pmatrix} g'_x & g'_y \end{pmatrix} = \begin{pmatrix} 2x & 2y \end{pmatrix}$$

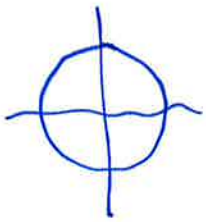
$$\boxed{\text{NDCQ: } \operatorname{rk} J = \underset{\uparrow\ \#\text{constraints}}{m} \text{ at all adm. pts.}}$$

In this case:

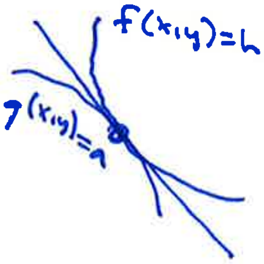
$\operatorname{rk} (2x\ \ 2y) = 1$

NDCQ fails:

$\left.\begin{array}{l} \operatorname{rk} (2x\ \ 2y) = 0 \\ \Updownarrow \\ 2x=0 \quad 2y=0 \\ (x,y) = (0,0) \leftarrow \text{not adm.} \end{array}\right\} \Rightarrow$ no adm. pts where NDCQ fails.



NDCQ holds.



$$\frac{f'_x}{g'_x} = \frac{f'_y}{g'_y}$$

What if $g'_x = g'_y = 0$?

Ex: $\qquad \overset{g_1}{x-y+2z} = \overset{a_1}{3}$ $\qquad$ Lagrange problem

$\qquad\quad \underset{g_2}{x+y} \quad = \underset{a_2}{3}$

NDCQ: $\quad J = \begin{pmatrix} \boxed{\begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix}} & \begin{matrix} 2 \\ 0 \end{matrix} \end{pmatrix}$

NDCQ condition: $\underline{\text{rk } J = 2}$ at all adm pts.

$\text{rk } J = 2$ for all (x, y, z)

NDCQ holds.

Kuhn-Tucker case: $\quad J = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots \\ \vdots & & \\ \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots \end{pmatrix}$

NDCQ: NDCQ holds for an adm. pt. $(x_1, \ldots, x_n)$ if the Lagrange NDCQ holds when you only include rows where the constraint is binding.

Ex: $\boxed{\begin{matrix} x-y+2z \geq 3 \\ x+y \quad \geq 3 \end{matrix}}$

a) $x-y+2z=3$
$\quad x+y \quad =3$

$\text{rk}\begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix} = 2$

ok.

b) $x-y+2z=3$
$\quad x+y \quad > 3$

$\text{rk}\,(1 \;\; -1 \;\; 2) = 1$

ok.

c) $x-y+2z>3$
$\quad x+y \quad =3$

$\text{rk}\,(1 \;\; 1 \;\; 0) = 1$

ok.

d) $x-y+2z>3$
$\quad x+y \quad >3$

no condition

Problem: max y when $x^2+y^3=0$

$$\mathcal{L} = y - \lambda \cdot (x^2+y^3)$$

i) **FOC:**

$$\boxed{\begin{aligned} \mathcal{L}'_x &= -\lambda \cdot 2x = 0 \\ \mathcal{L}'_y &= 1 - \lambda \cdot 3y^2 = 0 \end{aligned}}$$

C:

$$\boxed{x^2+y^3=0}$$

$\Rightarrow$ $\underline{\lambda=0}$ or $\underline{x=0}$

$\lambda=0$	$x=0$
$1-0\cdot 3y^2=0$	$x^2+y^3=0$
$1=0$	$0+y^3=0$
impossible	$y=0$
	$1-\lambda \cdot 3 \cdot 0^2=0$
	$1-\lambda\cdot 0=0$
	$1=0$
	impossible

$\Downarrow$

No solution of FOC+C = candidate pts

ii) **NDCQ fails:**

C: $x^2+y^3=0$

$$J = \begin{pmatrix} g'_x & g'_y \end{pmatrix} = \begin{pmatrix} 2x & 3y^2 \end{pmatrix}$$

$\text{rk}\, J = 0 \iff 2x=0,\ 3y^2=0$

$x=0,\ y=0$

$\Downarrow$

$\underline{(x,y)=(0,0)}$

adm. since $0^2+0^3=0$.

$\Downarrow$

One adm. pt. $\underline{(x,y)=(0,0)}$ where NDCQ fails.

$f(0,0)=\underline{0}$

$\Rightarrow$

Candidate pts:	none	$\leftarrow$ FOC+C
NDCQ fails:	$(0,0)$	$\underline{f=0}$

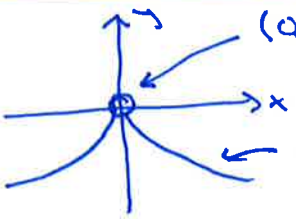
full list of all possible max. pts.

Is there a max?

$x^2+y^3=0$

$x^2=-y^3$

only possible if $y\le 0$



$(0,0)$ is the max pt.

$x^2+y^3=0$