Plenary Session 3

GRA 6035 BI

Eivind Eriksen, Oct 30 2017 — Mathematics

Plan:

1. Exam 01/2017, Pb. 3-4
2. Workbook: 7.1, 7.8/8.7, 7.11/8.10, 8.13
3. Exam 12/2015, Pb 3-4
4. Workbook: 8.8, 8.12

1. Exam 01/2017

Q3:

$$f = -3 - 2x^2 + 2xy - 2xz - 2y^2 + 4yz - 2z^2$$

$$\left.\begin{aligned} f'_x &= -4x + 2y - 2z \\ f'_y &= 2x - 4y + 4z \\ f'_z &= -2x + 4y - 4z \end{aligned}\right\} \quad H(f) = \begin{pmatrix} -4 & 2 & -2 \\ 2 & -4 & 4 \\ -2 & 4 & -4 \end{pmatrix}$$

a)

$$D_1 = -4$$
$$D_2 = 16 - 4 = 12$$
$$D_3 = -4(0) - 2 \cdot 0 - 2 \cdot 0 = 0$$

$$\Delta_1 = -4, -4, -4 < 0$$
$$\Delta_2 = 12, 0, 12 \geq 0$$
$$\Delta_3 = 0 \leq 0$$

Alt I: $\operatorname{rk} H(f) = 2$, $D_1 < 0, D_2 > 0$

$\boxed{\text{Thm}} \Rightarrow H(f)$ neg. semidefinite

$\Downarrow$

f concave, not convex

b) FOC:

$$f'_x = -4x + 2y - 2z = 0$$
$$f'_y = 2x - 4y + 4z = 0$$
$$f'_z = -2x + 4y - 4z = 0$$

$\Rightarrow (x,y,z) = (0,0,0)$ is a stationary pt.

$\Downarrow$

$f(0,0,0) = -3$ is max value of f.

$$\left(\begin{array}{ccc|c} -4 & 2 & -2 & 0 \\ 2 & -4 & 4 & 0 \\ -2 & 4 & -4 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 2 & -4 & 4 & 0 \\ 0 & -6 & 6 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

<u>Stationary pts:</u>

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ z \\ z \end{pmatrix}, \quad \underline{z \text{ free}}$$

$$\underline{-6y} + 6z = 0$$

$$y = \frac{-6z}{-6} = \underline{z}$$

$$2x - 4z + 4z = 0$$

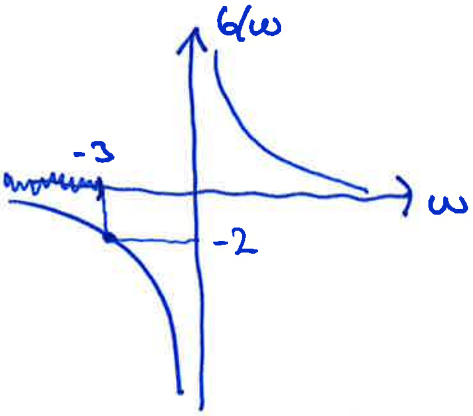
$$2x = 0$$

$$\underline{x = 0}$$

$$(x,y,z) = \underline{(0,z,z)}, \quad z \text{ free}$$

<u>Max:</u> $\underline{f(0,z,z) = -3}$ for all z

c) $$g(x,y,z) = \frac{6}{w} = \frac{6}{f(x,y,z)} = \frac{6}{-3-2x^2 \ldots -2z^2}$$



$$w \leq -3 \quad \begin{cases} f_{max} = -3 \\ f_{min} = (\to -\infty) \end{cases}$$

g_{max} : no max

g_{min} : -2

<u>Alt:</u>

$$g'(w) = (6/w)' = (6 \cdot w^{-1})' = 6 \cdot (-1)\, w^{-2} = \frac{-6}{w^2} < 0$$

g <u>decreasing function</u> : $w = -3$ <u>max</u> for f

$$\Downarrow$$

$g(w) = g(-3) = -2$ <u>min</u> for g

Ex: Solve $\min \sqrt{x^2+y^2}$ when $xy=1$

$(x,y)=(1,1)$

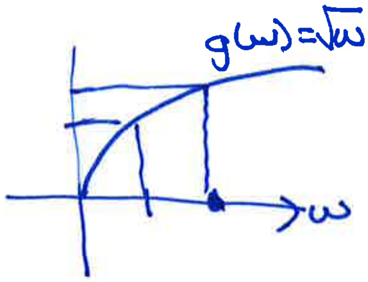
$\sqrt{z} \nearrow$

$\Downarrow$

$\min x^2+y^2$ when $xy=1$

$2 \nearrow$

$(x,y)=(1,1)$



$g'(w) = \frac{1}{2\sqrt{w}} > 0$

increasing function

Q 4. $\max f(x,y,z) = -3-2x^2+2xy-2xz-2y^2+4yz-2z^2$ when $x+y-z \geq 2$

$\Downarrow$

$-x-y+z \leq -2$

$L = f(x,y,z) - \lambda\cdot(-x-y+z)$

a) KT conditions:

FOC: $L'_x = -4x+2y-2z+\lambda = 0$

$L'_y = 2x-4y+4z+\lambda = 0$

$L'_z = -2x+4y-4z-\lambda = 0$

C: $x+y-z \geq 2$

CSC: $\lambda \geq 0$ and $\lambda\cdot(x+y-z-2)=0$

$\left.\begin{array}{l} \\ \end{array}\right\} \Leftrightarrow \left\{\begin{array}{ll} x+y-z=2,\ \lambda\geq 0 & \text{b)} \\ \text{or} & \\ x+y-z>2,\ \lambda=0 & \text{a)} \end{array}\right.$

b) Candidate pts:

a) $x+y-z>2,\ \lambda=0$

FOC: $\lambda=0 \Rightarrow \left.\begin{array}{l} f'_x=0 \\ f'_y=0 \\ f'_z=0 \end{array}\right\} \Rightarrow \underset{\displaystyle \underline{(0,z,z)}}{(x,y,z) \atop \shortparallel}$

$(x,y,z;\lambda)=(0,z,z;0)$

$x+y-z = 0+z-z=0 \not> 2$

no cand. pts

b) $x+y-z=2,\ \lambda\geq 0$

b)

$$\left.\begin{aligned} -4x + 2y - 2z + \lambda &= 0 \\ 2x - 4y + 4z + \lambda &= 0 \\ -2x + 4y - 4z - \lambda &= 0 \end{aligned}\right\} \text{FOC}$$

$$\left.\begin{aligned} x + y - z &= 2 \\ \lambda &\geq 0 \end{aligned}\right\} \text{C+CSC b)}$$

$$\begin{array}{c} \\ 4 \\ -2 \\ \\ \end{array}\left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 2 \\ -4 & 2 & -2 & 1 & 0 \\ 2 & -4 & 4 & 1 & 0 \\ \cancel{-2} & \cancel{4} & \cancel{-4} & \cancel{-1} & \cancel{0} \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 2 \\ 0 & 6 & -6 & 1 & 8 \\ 0 & -6 & 6 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{cccc|c} 1 & 1 & -1 & 0 & 2 \\ 0 & 6 & -6 & 1 & 8 \\ 0 & 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x = -(z+1) + z + 2 = \underline{1}$

$6y = 6z - 2 + 8 \Rightarrow y = \underline{z+1}$

$2\lambda = 4 \Rightarrow \underline{\lambda = 2}$

$(x,y,z;\lambda) = \underline{(1, z+1, z; 2)}$, z free

SOC: $L(x,y,z;2) = f(x,y,z) - 2\cdot(-x-y+z)$

$H(L) = H(f) = \begin{pmatrix} \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \end{pmatrix}$ concave from 3a).

$\Downarrow$

$(x,y,z) = (1, z+1, z)$ is max

Max value: $f(1, z+1, z) = f(1,1,0) = -3 - 2 + 2 - 2 = \underline{\underline{-5}}$

c) max $f(x,y,z)$ when $-1.12x - y + z \leq -2$:

max $f(x,y,z)$ when $cx - y + z + 2 \leq 0$

$L = f(x,y,z) - \lambda(cx - y + z + 2)$

$\boxed{c = -1: \text{ Problem b)}}$

$$\frac{df^*(c)}{dc} = L'_c(x^*(c), y^*(c), z^*(c); \lambda^*(c)) = (-\lambda x)(x^*(c), \ldots) = -\lambda^*(c)\cdot x^*(c)$$

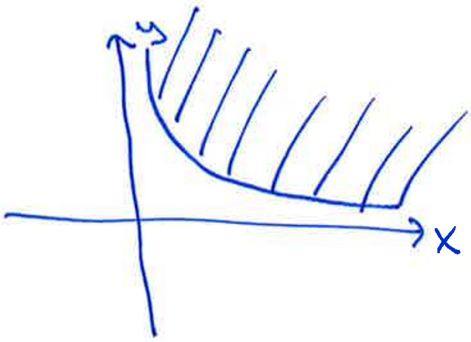
At $c=-1$: $\frac{df^*(c)}{dc} = -\lambda^*(-1) \cdot x^*(-1) = -2 \cdot 1 = \underline{-2}$

$$f^*(-1{,}12) \simeq \underset{-5}{\underset{4}{f^*(-1)}} + \underbrace{(-2)\cdot(-0.12)}_{0{,}24} = \underline{\underline{-4.76}}$$

Workbook.

7.1 v) $x \geq 0, y \geq 0, xy \geq 1$

* closed ($\geq$)
* $\boxed{\text{not bounded}}$

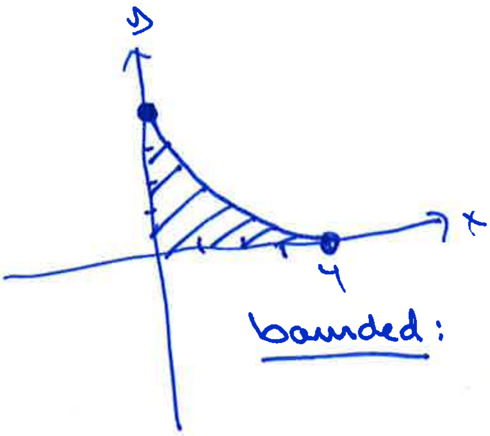


$xy \geq 1$:

$xy = 1$ (boundary) $\quad xy \geq 1$

$y = 1/x \quad y \geq 1/x$

vi) $\sqrt{x} + \sqrt{y} \leq 2$



bounded: $0 \leq x \leq 4$

$0 \leq y \leq 4$

$x;y \geq 0$

$\sqrt{x} + \sqrt{y} = 2$

$\sqrt{y} = 2 - \sqrt{x}$

$y = (2-\sqrt{x})^2$

$\sqrt{x} + \sqrt{y} \leq 2$

Exam 12/2015

3. $f(x,y,z) = \ln(u+1)$, $\quad u = 2x^2+2xy+3y^2-2xz+z^2$

a) $H(u) = \begin{pmatrix} 4 & 2 & -2 \\ 2 & 6 & 0 \\ -2 & 0 & 2 \end{pmatrix}$

$D_1 = 4 > 0$

$D_2 = 24-4 = 20 > 0$

$D_3 = -2\cdot 12 + 2\cdot 20 = 16 > 0$

u positive defn

$\Updownarrow$

$u(x,y,z) \geq 0$

$\Updownarrow$

$u+1 \geq 1$

$f(x,y,z) = \ln(u+1)$

defined since $\ln(w)$ defn. for $w>0$

FOC:

b) $f'_x = \frac{1}{u+1}\cdot (u+1)'_x = \frac{1}{u+1}\cdot(4x+2y-2z) = 0$

$f'_y = \qquad = \frac{1}{u+1}\cdot(2x+6y) = 0$

$f'_z = \qquad = \frac{1}{u+1}\cdot(-2x+2z) = 0$

$\Updownarrow$

$4x+2y-2z=0$

$2x+6y=0 \Rightarrow x = \underline{-3y}$

$-2x+2z=0 \Rightarrow x = \frac{-2z}{-2} = \underline{z}$

$\Updownarrow$

$x = z = -3y$

$4(-3y)+2y-2(-3y)=0$

$-12y+2y+6y=0$

$-4y=0$

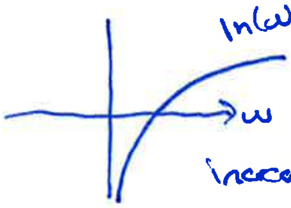
$\underline{y=0}$ $\quad \underline{x=0}$ $\quad \underline{z=0}$ $\quad \Rightarrow (x,y,z) = \underline{\underline{(0,0,0)}}$

$f(w) = \ln(1+w)$

$\Rightarrow f'(w) = \frac{1}{1+w} > 0$ incr. fn.

c) Min $f(x,y,z)$: $\quad f(0,0,0) = \ln(1) = \underline{0} \leftarrow$ unique crit pt.

$\ln(w) = \ln(u+1)$



min ~~value~~ point of $\ln(u+1)$ is the

min ~~value~~ point of u = $(0,0,0)$

$\Downarrow$

$f_{min} = \ln(1+u(0,0,0)) \underline{\underline{=0}}$ at $(0,0,0)$

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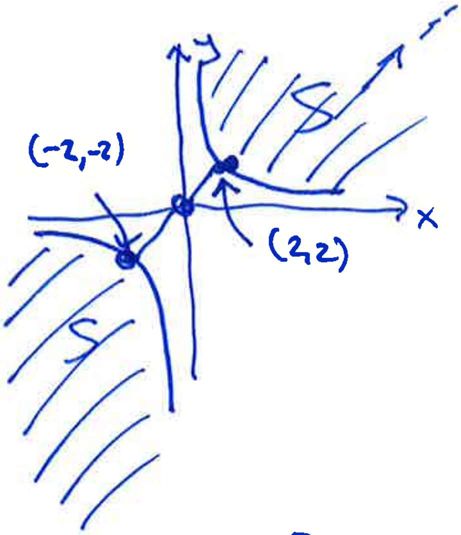
4. $\min f(x,y) = x^2+y^2$ when $xy \geq 4$

a) $S = \{(x,y) : xy \geq 4\}$ set of adm. pts.



$xy = 4$: $y = 4/x$

$xy \geq 4$: $xy > 4$

$y > 4/x \quad (x>0)$

$xy > 4$

$y < 4/x \quad (x<0)$

S is not bounded.

Explain why there is a minimum:

$$\boxed{\min\ x^2+y^2 \text{ when } xy \geq 4}$$

$$= \min \underbrace{\sqrt{x^2+y^2}}_{\text{distance from } (0,0) \text{ to } (x,y)} \text{ when } xy \geq 4 = \text{the point in } S \text{ closest to the origin } (0,0)$$

b) $\max \overset{-f}{\overbrace{-x^2-y^2}}$ when $-xy \leq -4$ $\boxed{\text{KT, std. form}}$

$L = -x^2-y^2 - \lambda\cdot(-xy)$

FOC: $L'_x = -2x + \lambda\cdot y = 0$

$L'_y = -2y + \lambda\cdot x = 0$

CSC+C: a) $xy = 4, \lambda \geq 0$:

$-2x + \lambda y = 0$

$-2y + \lambda x = 0$

$xy = 4$

$\Rightarrow x = \frac{\lambda y}{2}$

$-2y + \lambda\cdot\frac{\lambda y}{2} = 0 \mid \cdot 2$

$-4y + \lambda^2 y = 0$

$y(\lambda^2-4) = 0$

$\cancel{y=0}$ or $\lambda = \pm 2 = 2$

b) $xy > 4, \lambda = 0$:

$\lambda = 0 \Rightarrow x = y = 0$

$x\cdot y = 0 \not> 4$

$\boxed{\text{no cand. pts.}}$

$\underline{\lambda = 2:}$ $x = y$

$x^2 = 4$

$x = \pm 2$

$(x, y; \lambda) = (2, 2; 2)$ $f = 8$ $-f = -8$

$(-2, -2; 2)$ $f = 8$ $-f = -8$

min max

$\underline{\text{SOC:}}$ $\mathcal{L} = -x^2 - y^2 + \lambda(xy)$ ← Std. form

$= -x^2 - y^2 + 2xy$ ← $\underline{\text{concave?}}$

$H(\mathcal{L}) = \begin{pmatrix} -2 & 2 \\ 2 & -2 \end{pmatrix}$ $D_1 = -2$ $D_2 = 0$ $\Delta_1 = -2, -2$ $\Delta_2 = 0$

neg. semidefin. $\Rightarrow$ $\mathcal{L}$ concave $\Rightarrow$ $(x, y) = (2, 2)$

$(-2, -2)$

is $\underline{\text{max}}$

for $-f$

$= \underline{\text{min for } f}$

c) min $f = x^2 + y^2 + z^2 + w^2$ when $\begin{cases} xz \geq 9 \\ yw \geq 25 \end{cases}$

$= \max -f = -x^2 - y^2 - z^2 - w^2$ " $\begin{cases} -xz \leq -9 \\ -yw \leq -25 \end{cases}$ ← std. form

$\mathcal{L} = \underline{-x^2 - y^2 - z^2 - w^2 + \lambda_1 xz + \lambda_2 yw}$

$\mathcal{L}'_x = -2x + \lambda_1 z = 0$

$\mathcal{L}'_y = -2y + \lambda_2 w = 0$

$\mathcal{L}'_z = -2z + \lambda_1 x = 0$

$\mathcal{L}'_w = -2w + \lambda_2 y = 0$

$x, z, \lambda_1 \begin{cases} xz \geq 9 \\ \lambda_1 \geq 0 \end{cases}$ $\lambda_1 \cdot (xz - 9) = 0$

$y, w, \lambda_2 \begin{cases} yw \geq 25 \\ \lambda_2 \geq 0, \end{cases}$ $\lambda_2 (yw - 25) = 0$

x, z, λ_1:

$-2x + \lambda_1 z = 0 \qquad xz \geq 9$

$-2z + \lambda_1 \cancel{x} = 0 \qquad \lambda_1 \geq 0 \qquad \lambda_1(xz-9) = 0$

$xz > 9$: $\lambda_1 = 0$, $x = z = 0$ $\quad xz = 0 \not> 9 \Rightarrow$ no soln.

$xz = 9$: $x = \frac{\lambda_1 z}{2} \Rightarrow -2z + \lambda_1 \cdot \frac{\lambda_1 z}{2} = 0 \quad | \cdot 2$

$\lambda_1 > 0 \qquad x = z \qquad -4z + \lambda_1^2 \cdot z = 0$

$= \pm 3 \qquad z(\lambda_1^2 - 4) = 0$

$\lambda_1 = 2$

$\Downarrow$

$(x, z; \lambda_1) = (3, 3; 2)$
$(-3, -3; 2)$

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y, w, λ_2: the same way $\Rightarrow (y, w; \lambda_2) = (5, 5; 2)$
$(-5, -5; 2)$

$\Downarrow$

Candidates: $(x, y, z, w; \lambda_1, \lambda_2) = (3, 5, 3, 5; 2, 2) \qquad f = 2 \cdot 9 + 2 \cdot 25 = 68$
$(3, -5, 3, -5; 2, 2)$
$(-3, 5, -3, 5; 2, 2)$
$(-3, -5, -3, -5; 2, 2)$

concave?

SOC: $L(x, y, z, w, 2, 2) = -x^2 - y^2 - z^2 - w^2 + 2xz + 2yw$

$$H(L) = \begin{pmatrix} -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 2 & 0 & -2 & 0 \\ 0 & 2 & 0 & -2 \end{pmatrix}$$

note: rk = 2

$D_1 = -2$
$D_2 = 4$
$D_3 = 0$
$D_4 = 0$

thm: neg. semidefn.
$\Downarrow$
L concave
$\Downarrow$

$f = 68$ min