PLENARY SESSION 4

GRA 6035 MATHEMATICS

EIVIND ERIKSEN, NOV 13, 2017

Plan:

Workbook 10.8-2, 10.12.1, 10.15, 11.6.2, 11.11

CODES 1.14, 1.24, 2.3

Exam Jan 2017, 86.2

Workbook:

$y' = F(t,y)$

first order — linear / separable / exact

10.8.2.

$$(1+t^3)y' = t^2 y$$

$$y' = \frac{t^2 y}{1+t^3} = \frac{t^2}{1+t^3}\cdot y$$

$$\frac{1}{y}y' = \frac{t^2}{1+t^3}$$

$$\int \frac{1}{y}\,dy = \int \frac{t^2}{1+t^3}\,dt$$

$$\boxed{\begin{array}{c} u = 1+t^3 \\ du = 3t^2 \cdot dt \end{array}}$$

$$\ln|y| = \int \frac{\cancel{t^2}}{u}\cdot\frac{du}{3\cancel{t^2}} = \frac{1}{3}\int \frac{1}{u}\,du = \frac{1}{3}\ln|u| + C$$

$$\ln|y| = \frac{1}{3}\ln|1+t^3| + C$$

$$|y| = e^{\frac{1}{3}\cdot\ln|1+t^3| + C} = e^{\ln|1+t^3|^{1/3}}\cdot e^C$$

$$y = \pm e^C \cdot |1+t^3|^{1/3}$$

$$= \underline{\underline{K\cdot\sqrt[3]{1+t^3}}}$$

BI

10.12.1. $ty' + 2y + t = 0$

$ty' + 2y = -t$

$y' + \frac{2}{t}y = -1 \quad \cdot | \cdot t^2$ ← $y' + a(t)y = b(t)$ linear:

$\underbrace{t^2 y' + t^2 \frac{2}{t} y}= -t^2$

$(t^2 \cdot y)' = -t^2$

$t^2 \cdot y = \int -t^2 \, dt$

$\frac{t^2 \cdot y}{t^2} = \frac{-\frac{1}{3}t^3 + C}{t^2}$

$y = -\frac{1}{3}t + \frac{C}{t^2}$

Int. factor:

$\int a(t)\,dt = \int \frac{2}{t}\,dt$

$= 2\ln t + C$

$\Downarrow$

$u = e^{2\ln t} = (e^{\ln t})^2$ (t)

$= t^2$

Alt: $u = e^{3t}$

$(e^{3t} \cdot y)' = t \cdot e^{3t}$

$e^{3t} \cdot y = \int t \cdot e^{3t}\,dt$

Ex: $y' + 3y = t$ ← linear first order diff. eqn.

$a(t) = 3 \quad b(t) = t$

$y = y_h + y_p$ ← superposition principle

$= C \cdot e^{-3t} + \frac{1}{3}t - \frac{1}{9}$

$C \cdot e^{rt}$

y_h: $y' + 3y = 0$ homog.

$r + 3 = 0$ char. eqn.

$r = -3$ → $y_h = C \cdot e^{-3t}$

y_p: $y' + 3y = t$

$\cancel{A = t}$

$\cancel{A' = 1}$

$y = At + B$

$y' = A$

$A + 3(At + B) = t$

$\underset{1}{(3A)}t + \underset{0}{(A + 3B)} = t$

$A = 1/3$

$1/3 + 3B = 0 \Rightarrow B = -\frac{1}{9}$ $\Big\}$ $y_p = \frac{1}{3}t - \frac{1}{9}$

10.15.

$$\underbrace{2t+y}_{h'_t} + \underbrace{(t-4y)}_{h'_y} y' = 0 \quad , y(0)=0$$

$$h'_t + h'_y \cdot y' = 0 \quad \rightarrow \quad y' = \frac{-2t-y}{t-4y}$$

not sep. , not linear

$$h'_t = 2t+y \quad \Rightarrow \quad h = t^2 + yt + Q(y)$$

$$h'_y = t-4y$$

$$\Downarrow$$

$$h'_y = 0 + t + Q'(y)$$

$$= t - 4y$$

$$\Downarrow$$

$$Q'(y) = -4y$$

$$Q(y) = -2y^2$$

Concl:

the diff. eqn. is exact, with solution

$$h(t,y) = t^2 + yt - 2y^2 = C$$

$$-2y^2 + ty + (t^2 - C) = 0$$

$$y = \frac{-t \pm \sqrt{t^2 - 4\cdot(-2)(t^2-C)}}{2(-2)}$$

$$= \frac{t}{4} \pm \frac{1}{4}\sqrt{t^2 + 8(t^2-C)}$$

$$y = \frac{t}{4} \pm \frac{1}{4}\sqrt{9t^2 - 8C}$$

y(0)=0:

t=0,

y=0

$$0 = C$$

$$y(t) = \frac{t}{4} \pm \frac{1}{4}\sqrt{9t^2}$$

$$= \frac{t \pm 3t}{4}$$

$$y = t \quad \text{or} \quad y = -\frac{1}{2}t$$

11.6.2 $\quad 3y'' - 30y' + 75y = 2t+1$ $\qquad$ linear

BI

$$y = y_h + y_p = \underline{\underline{(C_1 + C_2 t)e^{5t} + \frac{1}{75}\left(2t + 37/25\right)}}$$

y_h: $\quad 3y'' - 30y' + 75y = 0$

$3r^2 - 30r + 75 = 0 \quad | : 3$

$r^2 - 10r + 25 = 0$

$(r-5)^2 = 0$

$r_1 = r_2 = 5 \longrightarrow y_h = \underline{C_1 \cdot e^{5t} + C_2 \cdot te^{5t}}$

y_p: $\quad 3y'' - 30y' + 75y = 2t+1$ $\qquad$ ~~$f = 2t+1$, $f' = 2$, $f'' = 0$~~ $\qquad$ $y = \underline{At+B}$, $y' = A$, $y'' = 0$

$3 \cdot 0 - 30 \cdot A + 75(At+B) = 2t+1$

$\underset{2}{\underset{\shortparallel}{(75A)}}t + \underset{1}{\underset{\shortparallel}{(75B - 30A)}} = 2t+1$

$\underline{A = 2/75}$ $\qquad$ $75B = 1 + 30 \cdot (2/75)$

$= 1 + 60/75$

$= 1 + 12/25 = 37/25$

$B = \frac{37}{25 \cdot 75}$

$$y_p = \frac{2}{75}t + \frac{37}{25 \cdot 75} = \underline{\frac{1}{75}\left(2t + 37/25\right)}$$

Exam: 01/2017, Pb 2

(a) $y'' - 3y' - 10y = 20$

$$y = y_h + y_p = \underline{\underline{c_1 e^{5t} + c_2 e^{-2t} \mp 2}}$$

y_h: $y'' - 3y' - 10y = 0$

$$r^2 - 3r - 10 = 0$$

$$r = \frac{3 \pm \sqrt{9+40}}{2}$$

$$= \frac{3 \pm 7}{2} = \underline{5}, \underline{-2} \rightarrow y_h = \underline{\underline{c_1 \cdot e^{5t} + c_2 \cdot e^{-2t}}}$$

$\underline{r_1 \neq r_2}$: $\quad y_h = c_1 \cdot \underline{e^{r_1 t}} + c_2 \cdot \underline{e^{r_2 t}}$

$\underline{r_1 = r_2}$: $\quad y_h = c_1 \cdot \underline{e^{r_1 t}} + c_2 \cdot \underline{t e^{r_1 t}}$

y_p: $y'' - 3y' - 10y = 20$

$\cancel{f = 20}$, $\cancel{f' = 0}$, $\cancel{f'' = 0}$

$\underline{y = A}$, $y' = 0$, $y'' = 0$

$$0 - 3 \cdot 0 - 10 \cdot A = 20$$

$$\underline{A = -2} \rightarrow \underline{y_p = -2}$$

b) $y' \cancel{+} \ln(t) = y \cdot \ln t$

$$y' = y \cdot \ln t - \ln t = (y-1) \cdot \ln t$$

$$y' - \ln t \cdot y = -\ln t$$

$$\left(y \cdot e^{-t\ln t + t}\right)' = -\ln t \cdot e^{-t\ln t + t}$$

$$y \cdot e^{-t\ln t + t} = \int -\ln t \cdot e^{-t\ln t + t} \, dt$$

$$= \int \cancel{-\ln t} \cdot e^{w} \cdot \frac{dw}{\cancel{-\ln t}}$$

$$\boxed{\begin{aligned} w &= -t\ln t + t \\ dw &= -\ln t \cdot dt \end{aligned}}$$

$$\int -\ln t \, dt$$

$$= \int \underset{u'}{-1} \cdot \underset{v}{\ln t} \, dt$$

$$= -t \cdot \ln t - \int -t \cdot \tfrac{1}{t} dt$$

$$= -t \cdot \ln t + \int 1 \, dt$$

$$= -t\ln t + t + C$$

$$\Downarrow$$

$$e^{-t\ln t + t}$$

$$= \int e^{w} dw = e^{w} + C$$

$$y e^{-t\ln t + t} = e^{-t\ln t + t} + C$$

$$y = \underline{\underline{1 + C \cdot e^{t\ln t - t}}}$$

c) $6t\,(y^2-t^2)^2 = 6y\,(y^2-t^2)^2\cdot y'\,,\; y(0)=1$

$6t = 6y\cdot y'$

$\int 6t\,dt = \int 6y\,dy$

$\frac{3t^2+C}{3} = \frac{3y^2}{3}$

$y^2 = t^2 + c/3$

$y = \pm\sqrt{t^2+c/3}$

$\underline{y(0)=1:}$ $\quad 1 = +\sqrt{0^2+c/3}$

$t=0, y=1$ $\quad c=3$

$\Downarrow$

$$y = \underline{\underline{\sqrt{t^2+1}}}$$

$\cancel{y^2=t^2 \Rightarrow y=t \text{ or } y=-t}$

$\underbrace{6t\cdot(y^2-t^2)^2}_{h'_t} - \underbrace{6y\,(y^2-t^2)^2\,y'}_{h'_y} = 0$

$\Downarrow$

$h = -(y^2-t^2)^3 = C$

$\left(\begin{array}{l} h'_t = -3\cdot(y^2-t^2)^2\cdot(-2t) \\ h'_y = -3\,(y^2-t^2)^2\cdot 2y \end{array}\right)$

$(y^2-t^2)^3 = -c$

$\underline{y(0)=1:}$ $\quad (1^2-0^2)^3 = -c$

$1 = -c$

$c = -1$

$(y^2-t^2)^3 = 1$

$y^2-t^2 = \sqrt[3]{1} = 1$

$y^2 = t^2+1$

$$y = \underline{\underline{\sqrt{t^2+1}}}$$

Differential Equations

1.24. ~~$y' = 1 - y^2$~~ Find all equilibrium states and determine their stability.

$$y' = y - y^2$$

$$y' = \underline{F(y)} \quad : \quad \underline{\text{autonomous}} \quad (\text{no } t \text{ in the equation})$$

Equilibrium state: $\underline{\underline{y_e = 0}}$, $\underline{\underline{y_e = 1}}$

$$F(y) = 0$$

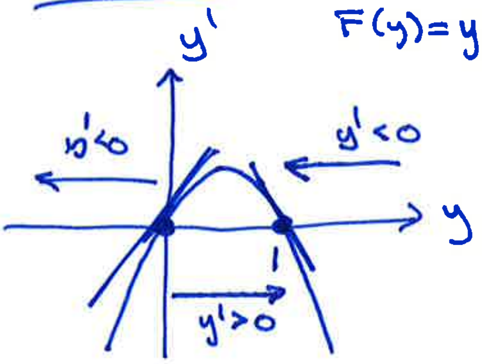
$$y - y^2 = 0$$

$$y(1-y) = 0$$

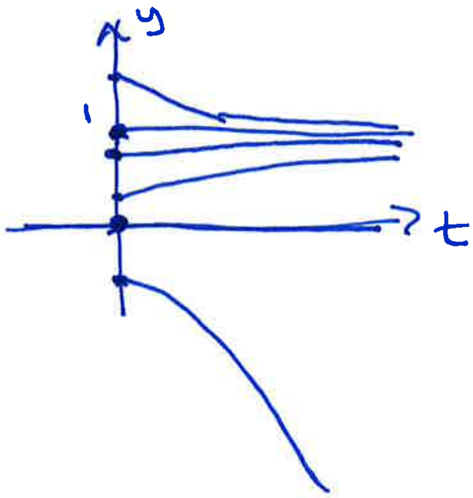
$\underline{y=0}$ or $\underline{y=1}$

Stability:

$F(y) = y - y^2$



phase diagram



$y_e = 1$ is $\underline{\underline{\text{stable}}}$

$y_e = 0$ is $\underline{\underline{\text{unstable}}}$

$$F'(y) = 1 - 2y$$

$F'(0) = 1 > 0$ $\underline{\text{unstable}}$

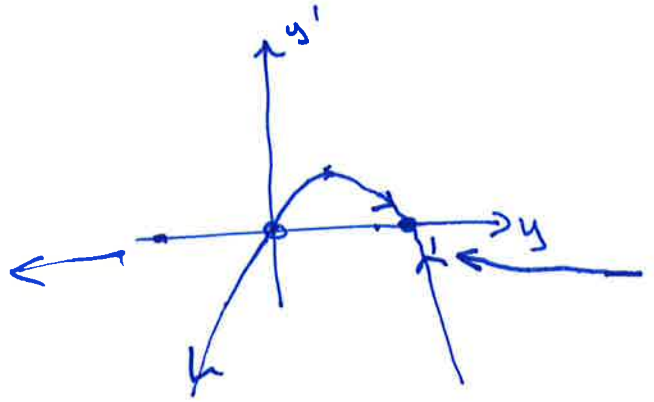
$F'(1) = -1 < 0$ $\underline{\text{stable}}$

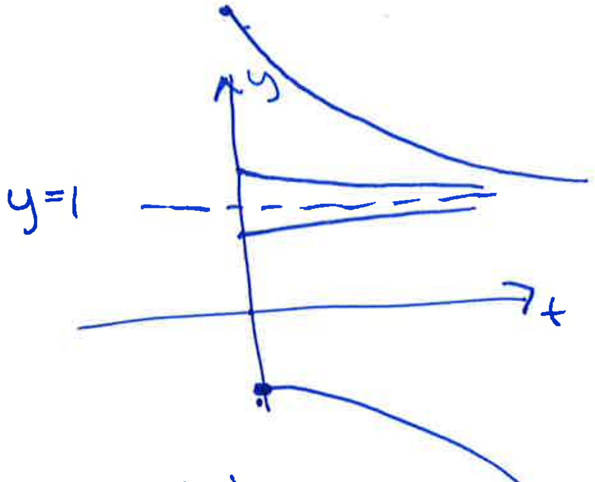
Stability Theorem:

y_e : eq. state for $y' = F(y)$

$F'(y_e) > 0 \implies y_e$ unstable

$F'(y_e) < 0 \implies y_e$ stable





$y_e = 1$ is <u>not</u> globally asymptotically stable because $y_0 < 0$ gives $y' < 0$ and y will decrease and not approach $y_e = 1$.

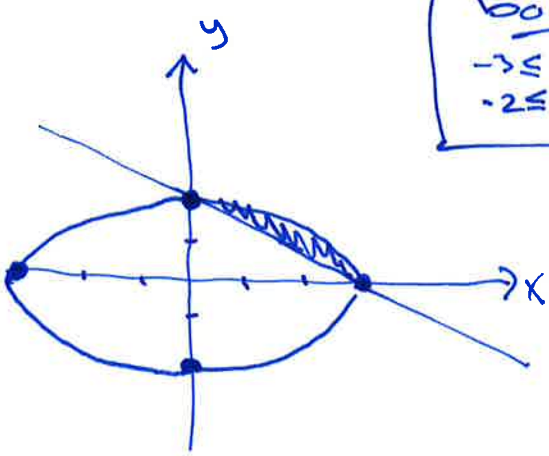
Exam 03/2016, Pb 4

max $f(x,y) = xy$ when $\begin{cases} 4x^2 + 9y^2 \le 36 \\ 2x + 3y \ge 6 \end{cases}$

$\Updownarrow$

$-2x - 3y \le -6$

a) $S = \{(x,y) : 4x^2 + 9y^2 \le 36,\ 2x + 3y \ge 6\}$



$\boxed{\begin{array}{c} \text{bounded} \\ -3 \le x \le 3 \\ -2 \le y \le 2 \end{array}}$

① $4x^2 + 9y^2 = 36$: ellipse

$x = 0$: $9y^2 = 36$ $\quad y^2 = 4$

$y = 0$: $4x^2 = 36$ $\quad x^2 = 9$

$4x^2 + 9y^2 < 36$: inside

② $2x + 3y = 6$: line $\quad y = \frac{6-2x}{3} = 2 - \frac{2}{3}x$

$x = 0$: $y = 2$

$y = 0$: $x = 3$

$2x + 3y > 6$: upper side

b) Find candidate pts with $\lambda_1 = 1/12$

$$L = xy - \lambda_1(4x^2+9y^2) + \lambda_2(2x+3y)$$

$\boxed{\lambda_1 = 1/12}$

$\lambda_1 = 1/12$

FOC $\begin{cases} L'_x = y - \lambda_1 \cdot 8x + 2\lambda_2 = 0 \\ L'_y = x - \lambda_1 \cdot 18y + 3\lambda_2 = 0 \end{cases}$

C: $\begin{cases} 4x^2+9y^2 \leq 36 \\ 2x+3y \geq 6 \end{cases}$

ok.

CSC: $\begin{cases} \lambda_1 \geq 0, \ \lambda_1 \cdot (4x^2+9y^2-36) = 0 \\ \lambda_2 \geq 0, \ \lambda_2 \cdot (2x+3y-6) = 0 \end{cases}$

$4x^2+9y^2 = 36$

$2x+3y = 6$, $\lambda_2 \geq 0$

$y - \frac{2}{3}x + 2\lambda_2 = 0 \quad \cdot 3$

$x - \frac{3}{2}y + 3\lambda_2 = 0 \quad \cdot 2$

$\cancel{(3y+2x)} - \cancel{2x - 3y} + 12\lambda_2 = 0$

$\lambda_2 = 0$

$\Downarrow$

$y = \frac{2}{3}x$: $2x + 3 \cdot \frac{2}{3}x = 6$

$2x + 2x = 6$

$x = 6/4 = 3/2$

$y = 1$

$4\left(\frac{3}{2}\right)^2 + 9 \cdot 1^2 = 9+9 = 18 \neq 36$

no candidate pts

$2x+3y > 6$

$\lambda_2 = 0$: $y - \frac{8}{12}x = 0 \Rightarrow y = \frac{2}{3}x$

~~$x - \frac{18}{12}y = 0 \quad x = \frac{3}{2}y$~~

$4x^2 + 9\left(\frac{2}{3}x\right)^2 = 36$

$4x^2 + 4x^2 = 36$

$8x^2 = 36$

$x^2 = 36/8 = 18/4$

$x = \sqrt{18}/2$ or ~~$x = -\sqrt{18}/2$~~

$y = \sqrt{18}/3$ ~~$x = -\sqrt{18}/3$~~

$2x+3y = \sqrt{18} + \sqrt{18}$ ✓

$\Downarrow$

$\boxed{\text{One candidate: } x = \sqrt{18}/2 \quad y = \sqrt{18}/3 \quad \lambda_1 = \frac{1}{12} \quad \lambda_2 = 0}$

c) Check $(\sqrt{18}/2, \sqrt{18}/3 ; 1/12, 0)$ using SOC:

$$L(x,y;\tfrac{1}{12},0) = xy - \tfrac{1}{12}(4x^2+9y^2)$$

$$L'_x = y - \tfrac{1}{12}\cdot 8x = y - \tfrac{2}{3}x$$

$$L'_y = x - \tfrac{1}{12}\cdot 18y = x - \tfrac{3}{2}y$$

$$H = \begin{pmatrix} -2/3 & 1 \\ 1 & -3/2 \end{pmatrix}$$

$D_1 = -2/3 < 0$ $\quad \Delta_1 = -3/2 < 0$

$D_2 = 1 - 1 = 0$

$\Downarrow$

H neg. semidefin

$\Downarrow$

$$(x,y) = \left(\frac{\sqrt{18}}{2}, \frac{\sqrt{18}}{3}\right) \text{ is max} \xLeftarrow[\text{SOC}]{} L(x,y;\tfrac{1}{12},0) \text{ concave}$$

Max value: $f = xy = \frac{18}{6} = \underline{\underline{3}}$