Plan:

Review: Lecture 10

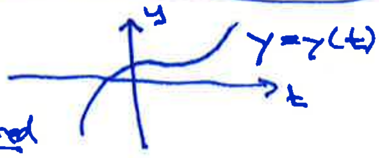
1. Exact differential equations
2. Stability of equilibrium states
3. Second order differential equations

Review:

First order differential equations $y' = F(t,y)$

- general solution: depends on one undetermined coeff.
- particular solution: one initial condition $\Rightarrow$ unique solution

$\boxed{y = \text{expression in } t}$ explicit form



most cases not covered

Cases: – cases not mutually exclusive

(a) **Separable:**

$y' = f(y)\cdot g(t)$

Method: Separation of variables

$\frac{1}{f(y)}\cdot y' = g(t) \Rightarrow \boxed{\int \frac{1}{f(y)}\,dy = \int g(t)\,dt}$

Solve integrals, and then solve for y.

(b) **Linear:**

$y' + a(t)\cdot y = b(t)$

$\Updownarrow$

$y' = b(t) - a(t)\cdot y$

Method: Integrating factor

Int. factor: $u(t) = e^{\int a(t)\,dt}$ ← may put $c=0$

$(y\cdot u)' = b(t)\cdot u(t)$

$\Updownarrow$

$y\cdot u = \int b(t)\,u(t)\,dt$

$\Updownarrow$

$\boxed{y = \frac{1}{u(t)}\cdot \int b(t)\,u(t)\,dt}$

Alternative method: Superposition

(when $a(t) = a$ is constant)

(c) Exact: To be covered today, see next page.

Ex: $y' + 2ty = 2t$ first order diff. eqn.

$y' = 2t - 2ty$

(a) Separable: Yes.

$y' = 2t \cdot (1-y) \Rightarrow \frac{1}{1-y} y' = 2t$

$$\int \frac{1}{1-y} dy = \int 2t \, dt$$

$$-\ln|1-y| = t^2 + C$$

$$e^{\ln|1-y|} = e^{-t^2 - c}$$

$$|1-y| = e^{-t^2} \cdot e^{-c}$$

$y = \underline{\underline{1 - Ke^{-t^2}}}$ $\Leftarrow$ $1 - y = (\pm e^{-c}) e^{-t^2} = Ke^{-t^2}$

general solution

(b) Linear: Yes.

$y' + \underbrace{2t}_{a(t)} y = \underbrace{2t}_{b(t)}$

$\int 2t \, dt = t^2 + C \Rightarrow \underline{u = e^{t^2}}$

$$(y \cdot e^{t^2})' = 2t \cdot e^{t^2}$$

$$y \cdot e^{t^2} = \int 2t e^{t^2} dt = \int \cancel{2t} e^u \cdot \frac{du}{\cancel{2t}}$$

$\boxed{\begin{matrix} u = t^2 \\ du = 2t\,dt \end{matrix}}$

$$ye^{t^2} = \int e^u du = e^u + C$$

$$y = \frac{ye^{t^2}}{e^{t^2}} = \frac{e^{t^2} + C}{e^{t^2}} = \underline{\underline{1 + Ce^{-t^2}}}$$

general solution

1. Exact differential equations

Defn: A first order differential equation is exact if it can be written

$$p(t,y) + q(t,y)\cdot y' = 0$$

where p, q are functions such that

$$\boxed{p = h'_t \quad \text{and} \quad q = h'_y}$$

for a function $h = h(t,y)$ in two variables.

Ex: $\underbrace{y}_{p} + \underbrace{t}_{q}y' = 0$

Is this exact?

$\Updownarrow$

Is there a fn. $h(t,y)$ such that

$$\left.\begin{aligned} p &= y = h'_t \\ q &= t = h'_y \end{aligned}\right\}$$

Try to find h:

I $\quad h'_t = y$

II $\quad h'_y = t$

I: $h'_t = y$

$\Downarrow$

$h = \int y \, dt = yt + g(y)$

II: $(yt + g(y))'_y = t$

$t + g'(y) = t$

$g'(y) = 0$

Choose $g(y) = 0$.

$\Downarrow$

$h(t,y) = yt$ is one solution

Yes, $y + ty' = 0$ is exact

General solution:

$h(t,y) = C$

$yt = C \Rightarrow y = C/t$

Solution method for exact differential equations:

If there is a fn. $h(t,y)$ such that $h'_t = p$ and $h'_y = q$, then the general solution is

$$\boxed{h(t,y) = C}$$

Why?

$$h'_t + h'_y \cdot y' = 0$$

$$\underbrace{\frac{\partial h}{\partial t} + \frac{\partial h}{\partial y} \cdot \frac{dy}{dt}}_{} = 0$$

$$\frac{dh}{dt} = 0$$

$$h = C$$

Ex: $\underbrace{1 + ty^2}_{p} + \underbrace{t^2 y}_{q} \cdot y' = 0$

Level curves:

$f(x,y) = a$

$f'_x + f'_y \cdot y' = 0$

$y' = -\dfrac{f'_x}{f'_y}$

Exact?
$$\boxed{\begin{aligned} h'_t &= 1 + ty^2 \\ h'_y &= t^2 y \end{aligned}}$$

$$h'_t = 1 + ty^2 \Rightarrow h = \int 1 + ty^2 \, dt$$
$$= t + y^2 \cdot \tfrac{1}{2} t^2 + g(y)$$

$$\left(t + \tfrac{1}{2} y^2 t^2 + g(y)\right)'_y = t^2 y$$

$$\cancel{0} + y \cdot t^2 + \underset{0}{\underset{\shortparallel}{g'(y)}} = t^2 y \Rightarrow$$

diff. eqn. is exact

Can choose $g(y) = 0$

$$h = t + \tfrac{1}{2} t^2 y^2 = C$$

$$\frac{\tfrac{1}{2} t^2 y^2}{\tfrac{1}{2} t^2} = \frac{C - t}{\tfrac{1}{2} t^2} \quad \begin{matrix} \cdot 2 \\ \cdot 2 \end{matrix}$$

$$y^2 = \frac{2(C-t)}{t^2}$$

$$y = \pm \sqrt{\frac{2(C-t)}{t^2}}$$

Result: Condition for exactness

$p + q \cdot y' = 0$ is exact $\iff$ $p'_y = q'_t$

Note that this can be used to check exactness, but not to solve the diff. eqn if it is exact.

To solve it, ~~we~~ must find $h(t, y)$ explicitly.

Explanation: Total derivative

If $y = y(t)$ and $h = h(t, y)$, then a small change in t will give

$$h(t + \Delta t, y(t + \Delta t)) \simeq \cancel{h(t + \Delta t, y(t))}$$

$$\simeq h(t + \Delta t, y(t) + y' \cdot \Delta t)$$

$$= h(t, y(t)) + h'_t \cdot \Delta t + h'_y \cdot y' \cdot \Delta t$$

$$\Rightarrow \frac{\Delta h}{\Delta t} \approx \frac{h(t + \Delta t, y(t + \Delta t)) - h(t, y(t))}{\Delta t}$$

$$= \underline{h'_t + h'_y \cdot y'} \quad \leftarrow \text{total derivative of } h \text{ w.r.t. } t.$$

2 Equilibrium states and stability

$y' = F(y)$ autonomous case

$$\boxed{\text{Equilibrium states: } y' = 0 \Leftrightarrow F(y) = 0}$$

Ex. $y' = 1 - 2y$

$1 - 2y = 0$

$y_e = 1/2$

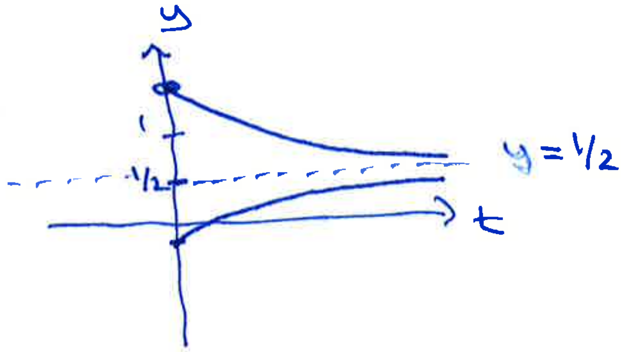
If we start at the equilibrium state y_e, y will stay there.

Solve $y' = 1 - 2y$:

$\Downarrow$

$y = Ce^{-2t} + \frac{1}{2}$

$y_e = 1/2$ is stable



Defn:

stable: $y_0 \neq y_e$ but close to y_e
$\Downarrow$
$y(t) \to y_e$ as $t \to \infty$

unstable: $y_0 \neq y_e$ but close to y_e
$\Downarrow$
$y(t)$ moves away from y_e as $t \to \infty$

$t = 0$: $y_0 = C \cdot e^0 + \frac{1}{2} = C + \frac{1}{2}$

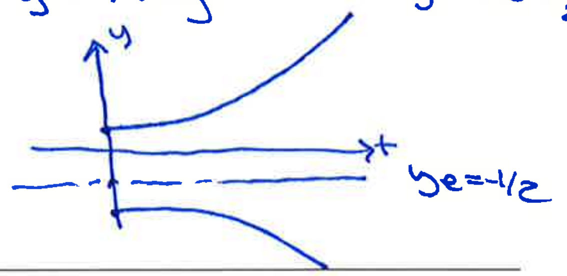
$y_0 = y(0)$

$C = y_0 - \frac{1}{2}$

$$\boxed{e^{-2t} \to 0 \text{ as } t \to \infty}$$

If $y' = 1 + 2y$ $\quad y = Ce^{2t} - \frac{1}{2}$

$y_e = -\frac{1}{2}$ unstable



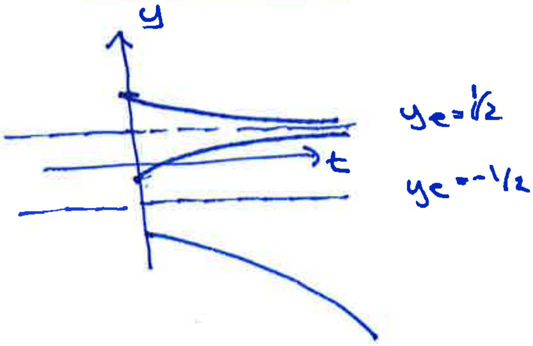
Stable
globally as. stable

unstable

Stable,
not globally
as. stable

How to find the stability of an equilibrium state?

Ex! $y' = 2 - 8y^2$



Eq. state: $2 - 8y^2 = 0$

$\frac{8y^2}{8} = \frac{2}{8}$

$y^2 = 1/4$

$y = \pm 1/2$

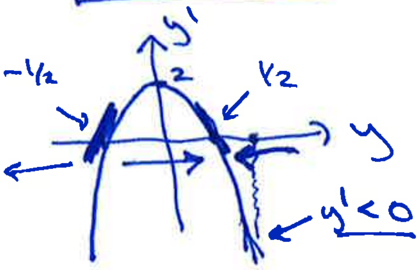
$y_e = 1/2$

$y_e = -1/2$

$y_e = \frac{1}{2}$ Stable, not globally as. stable

$y_e = -\frac{1}{2}$ unstable

phase diagram:



$\boxed{\text{ye globally asymptotically stable: for any } y_0,\ y(t) \text{ will converge towards ye as } t \to \infty}$

Stability theorem:
$y' = F(y)$ has eq. state ye
$F'(y_e) < 0 \Rightarrow$ ye is stable
$F'(y_e) > 0 \Rightarrow$ ye is unstable

$F = 2 - 8y^2$
$F' = -16y$
$F'(1/2) = -8$
$F'(-1/2) = 8$

(3) Second order differential equations

Differential equations involving y''.

Ex: $y'' = 6$

$y' = \int 6\,dt = 6t + C$

$y = \int 6t + C\,dt = \underline{3t^2 + Ct + D}$

general solution: two undetermined coeff. C, D (for all second order diff. eqn.)

$y'' = 6$, $y(0) = 3$, $y'(0) = 5$

$\Downarrow$

$y = 3t^2 + Ct + D$ $\quad$ $y(0)=3$: $3 = 3 0^2 + C\cdot 0 + D \Rightarrow D = 3$

$y' = 6t + C$ $\quad$ $y'(0) = 5$: $5 = 6\cdot 0 + C \Rightarrow C = 5$

$\Downarrow$

Part. sol'n: $y = 3t^2 + 5t + 3$

Case: Linear second order differential equations with constant coeff's.

$$\boxed{y'' + ay' + by = f(t)}$$

a, b: constants

f(t): expr. in t

(a) Homogeneous case: $f(t) = 0$

$y'' + ay' + by = 0$

Characteristic equation: $\begin{cases} y'' \to r^2 \\ y' \to r \end{cases}$ $\quad y \to 1$

$r^2 + ar + b = 0$

$\Downarrow$

$r = \dfrac{-a \pm \sqrt{a^2 - 4b}}{2}$ characteristic roots

i) $a^2-4b>0$: Two distinct roots $r_1 \neq r_2$

General sol'n: $y = C_1 \cdot e^{r_1 t} + C_2 \cdot e^{r_2 t}$

ii) $a^2-4b=0$: One root r of mult. 2

General sol'n: $y = C_1 \cdot e^{rt} + C_2 \cdot te^{rt}$

iii) $a^2-4b<0$: No real solutions (roots)

$$r = \frac{-a \pm \sqrt{a^2-4b}}{2} = \underbrace{\frac{-a}{2}}_{\alpha} \pm \underbrace{\frac{\sqrt{4b-a^2}}{2}}_{\beta} \cdot \sqrt{-1}$$

$$\alpha = \frac{-a}{2}$$

$$\beta = \sqrt{4b-a^2}/2$$

General solution: $y = e^{\alpha t}\left(C_1 \cdot \cos(\beta t) + C_2 \sin(\beta t)\right)$

Ex: $y'' - 3y' + 2y = 0$

$r^2 - 3r + 2 = 0$

$r = 1$, $r = 2$

$$r = \frac{3 \pm \sqrt{9-8}}{2} \Rightarrow y = C_1 e^{t} + C_2 e^{2t}$$

$y'' - 4y' + 4y = 0$

$r^2 - 4r + 4 = 0$

$$r = \frac{4 \pm \sqrt{16-16}}{2}$$

$r_1 = r_2 = 2 \Rightarrow y = C_1 e^{2t} + C_2 \cdot te^{2t}$

(b) General case: $y'' + ay' + by = f(t)$

Superposition principle:

$y = y_h + y_p$, where

general solution

y_h: homogeneous solution = general solution of the homogeneous eqn. $y'' + ay' + by = 0$

y_p: particular solution of $y'' + ay' + by = f(t)$

works also for first order linear diff eqn. with const. coeffs.: $y' + ay = f(t)$

Ex: $y'' - 3y' + 2y = e^{2t}$

$$y = y_h + y_p = \underbrace{C_1 e^t + C_2 e^{2t}}_{y_h} + \underbrace{te^{2t}}_{y_p}$$

y_h: $y'' - 3y' + 2y = 0$

$r^2 - 3r + 2 = 0$

$r = 1,\ r = 2 \implies y_h = C_1 e^t + C_2 e^{2t}$

y_p: $y'' - 3y' + 2y = e^{2t}$

Guess: $y = A \cdot e^{2t}$

$y' = A \cdot 2e^{2t}$

$y'' = A \cdot 4e^{2t}$

$(4A\cancel{e^{2t}}) - 3(2A\cancel{e^{2t}}) + 2(A\cancel{e^{2t}}) = \cancel{e^{2t}}$

$4A - 6A + 2A = 1$

$0 \cdot A = 1$

Method of undetermined coeffs.:

1. Start with $f(t) = e^{2t}$, compute $f'(t) = 2e^{2t}$, $f''(t) = 4e^{2t}$
2. Choose a guess $y(t)$ s.t.
   i) f, f', f'' are special cases
   ii) with undetermined coeffs.
   $y = A \cdot e^{2t}$
3. Check if $y(t)$ fits for any value of A.

Two poss.

- no solution: $0 \cdot A = 1$ $\Downarrow$ try to multiply the guess with t: $\boxed{y = Ate^{2t}}$ Repeat the steps.
- there is a solution for A. $\Rightarrow$ $y_p = Ae^{2t}$

In this case: $y = Ae^{2t}$ doesn't work : $0 \cdot A = 1$ no soln for A

New guess: $y = Ate^{2t}$

$$y' = A \cdot e^{2t} + Ate^{2t} \cdot 2 = (A + 2At)e^{2t}$$

$$y'' = 2Ae^{2t} + (A + 2At)e^{2t} \cdot 2 = (4A + 4At)e^{2t}$$

try in diff. eqn.

$$y'' - 3y' + 2y = e^{2t}$$

$$(4A + 4At)e^{2t} - 3(A + 2At)e^{2t} + 2(Ate^{2t}) = e^{2t}$$

$$(4A - 3A + \underbrace{4At - 6At + 2At}_{=0})e^{2t} = e^{2t}$$

$$\frac{Ae^{2t}}{e^{2t}} = \frac{e^{2t}}{e^{2t}} \qquad A = 1 \Rightarrow y_p = 1 \cdot te^{2t} = te^{2t}$$

$$y = y_h + y_p = C_1e^t + C_2e^{2t} + te^{2t}$$

A simpler example:

Ex: $y'' - 3y' + 2y = e^{-t}$

$y = y_h + y_p = C_1 e^{t} + C_2 e^{2t} + \frac{1}{6} e^{-t}$

y_h: $y'' - 3y' + 2y = 0$ (same as previous example)

$y = C_1 e^{t} + C_2 e^{2t}$

y_p: $f(t) = e^{-t}$ $\Rightarrow$ Guess: $y = Ae^{-t}$

$f'(t) = -e^{-t}$ $y' = -Ae^{-t}$

$f''(t) = +e^{-t}$ $y'' = Ae^{-t}$

$\Downarrow$

$y'' - 3y' + 2y = e^{-t}$

$(Ae^{-t}) - 3(-Ae^{-t}) + 2(Ae^{-t}) = e^{-t}$

$(A + 3A + 2A)e^{-t} = e^{-t}$

$6A \cdot e^{-t} = e^{-t}$

$\Downarrow$

$6A = 1$ $A = 1/6$

In this case, we found y_p with our first guess $\longrightarrow$ $y_p = \frac{1}{6} e^{-t}$