Plan:

Review of Lecture 2

1. Vectors and span
2. Linear independence of vectors
3. Rank and linear independence

Review of Lecture 2:

- matrix multiplication and the identity matrix I
- determinants and minors
- inverse matrices: $\boxed{A \text{ invertible} \Leftrightarrow |A| \neq 0 \\ (A^{-1} \text{ exists})}$
- rank, minors and linear systems: $\operatorname{rk} A$ = maximal order of a non-zero minor.

Ex:

$$\hat{A} = (A|\underline{b}) = \begin{array}{c|c} \begin{pmatrix} -1 & 3 & 1 & 2 & 4 \\ 2 & 0 & 2 & -1 & 3 \\ -4 & 9 & 1 & 7 & 9 \end{pmatrix} & \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \\ \begin{matrix} 1 & 2 & 3 & 4 & 5 \end{matrix} & \end{array}$$

$$M_{123,123} = \begin{vmatrix} -1 & 3 & 1 \\ 2 & 0 & 2 \\ -4 & 9 & 1 \end{vmatrix} = -2 \cdot (-6) - 2 \cdot 3 = 6 \neq 0$$

$$\Downarrow$$

$$\underline{\operatorname{rk} A = 3}$$

$$M_{123,345} = 1 \cdot (-30) - 2 \cdot 15 + 4 \cdot 15 = \underline{0}$$

Gauss:

x_1, x_2, x_3: basic
x_4: free

infinitely many solutions

$$(A|\underline{b}) = \left(\begin{array}{ccc|c|c} -1 & 3 & 1 & 2 & 4 \\ 2 & 0 & 2 & -1 & 3 \\ -4 & 9 & 1 & 7 & 9 \end{array}\right)$$

$$\underbrace{x_1 \quad x_2 \quad x_3}_{\text{basic}} \quad \underbrace{x_4}_{\text{free}}$$

det( )

$M_{123,123} \neq 0$

$\Downarrow$

Ignore eqn's that does not go through the box.

Variables x_1, x_2, x_3 are basic, the others are free.

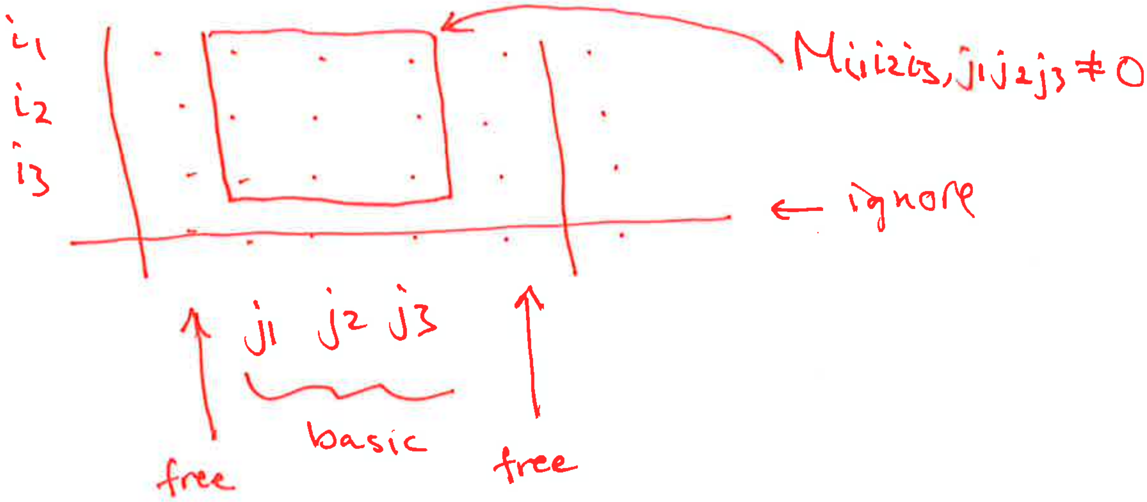
$$\begin{pmatrix} -1 & 3 & 1 \\ 2 & 0 & 2 \\ -4 & 9 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 - 2x_4 \\ 3 + x_4 \\ 9 - 7x_4 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 & 3 & 1 \\ 2 & 0 & 2 \\ -4 & 9 & 1 \end{pmatrix}^{-1} \cdot \begin{pmatrix} 4 - 2x_4 \\ 3 + x_4 \\ 9 - 7x_4 \end{pmatrix}$$

In general:

If $\operatorname{rk} A = \operatorname{rk}(A|\underline{b}) = r$ and $M_{i_1 i_2 \ldots i_r, j_1 j_2 \ldots j_r} \neq 0$, then:

- ignore all eqn's except $i_1, i_2, \ldots, i_r$
- all variables except $x_{j_1}, x_{j_2}, \ldots, x_{j_r}$ are free
- it is possible to solve for $x_{j_1}, x_{j_2}, \ldots, x_{j_r}$ in terms of free variables (i.e. $x_{j_1} \ldots x_{j_r}$ are basic)



① Vectors and span

A **vector** (m-vector) is an $m \times 1$-matrix, and it is sometimes called column vectors.

vectors are underlined →

$$\underline{v} = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \qquad \underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \qquad \underline{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \qquad \underline{b} = \begin{pmatrix} 1 \\ 2 \\ 7 \end{pmatrix}$$

Ex: $A \cdot \underline{x} = r \cdot \underline{x}$ $\left\{ \begin{array}{l} A: \text{matrix} \\ \underline{x}: \text{vectors} \\ r: \text{number} \end{array} \right.$

Linear combinations: using addition and scalar multiplication

$\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$: vectors of the same size

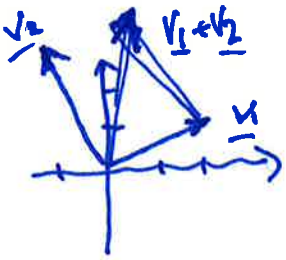
$r_1, r_2, \ldots, r_n$: numbers

$$\boxed{r_1 \cdot \underline{v}_1 + r_2 \cdot \underline{v}_2 + \ldots + r_n \cdot \underline{v}_n}$$
: linear combination of $\underline{v}_1, \ldots, \underline{v}_n$, a new vector of the same size

Ex: $\underline{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} -1 \\ 3 \end{pmatrix}$: $r_1 \cdot \underline{v}_1 + r_2 \cdot \underline{v}_2 = r_1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + r_2 \begin{pmatrix} -1 \\ 3 \end{pmatrix}$

$$= \begin{pmatrix} 2r_1 \\ r_1 \end{pmatrix} + \begin{pmatrix} -r_2 \\ 3r_2 \end{pmatrix}$$

$$= \begin{pmatrix} 2r_1 - r_2 \\ r + 3r_2 \end{pmatrix}$$

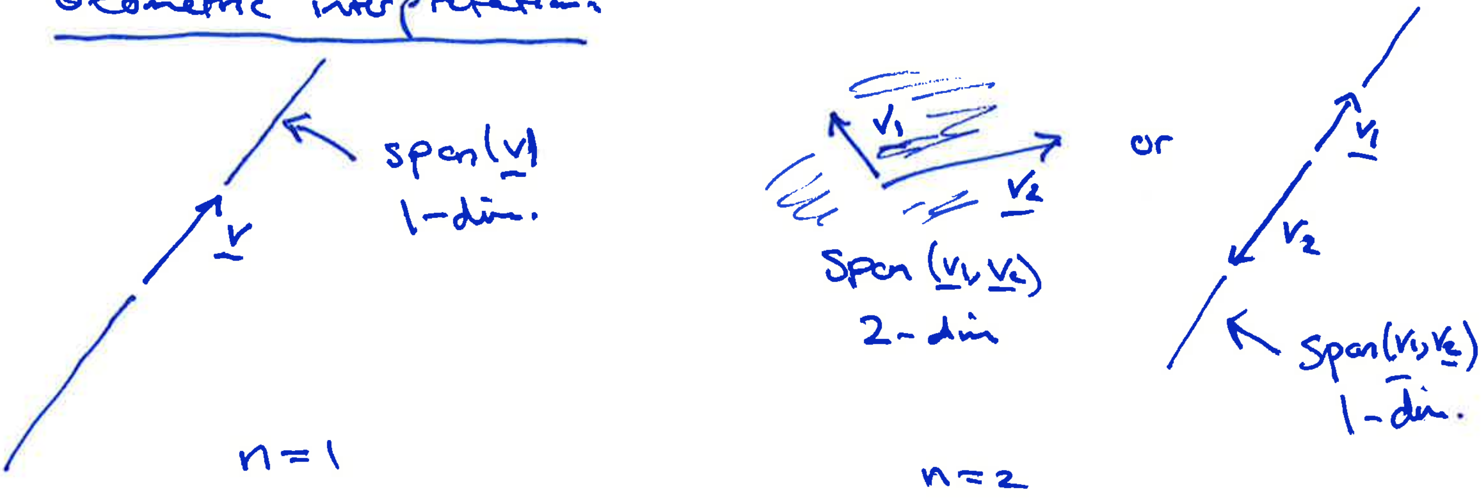


Ex: $2 \cdot \underline{v}_1 + 3 \underline{v}_2 = 2 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 3 \begin{pmatrix} -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 11 \end{pmatrix}$

<u>Defn:</u> $\text{span}(\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n) = \{ r_1 \cdot \underline{v}_1 + r_2 \underline{v}_2 + \dots + r_n \underline{v}_n : r_1, r_2, \dots, r_n \}$

= set of all linear combinations of $\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n$

<u>Geometric interpretation</u>



<u>Ex:</u> $\underline{v}_2 = -2\underline{v}_1$ $\boxed{\text{span}(\underline{v}_1, \underline{v}_2) = \text{span}(\underline{v}_1)}$

$\underline{v}$ is in span$(\underline{v}_1, \underline{v}_2)$: $\underline{v} = r_1 \cdot \underline{v}_1 + r_2 \cdot \underline{v}_2$

$= r_1 \cdot \underline{v}_1 + r_2 \cdot (-2\underline{v}_1)$

$= (r_1 - 2r_2)\underline{v}_1$ $\underline{v}_1$ is in span$(\underline{v}_1)$

Ex: $\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ $\quad \underline{v}_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$ $\quad$ $\text{span}(\underline{v}_1, \underline{v}_2) = ?$

Is $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$ in $\text{span}(\underline{v}_1, \underline{v}_2)$? Yes.

$$\begin{pmatrix} 5 \\ 2 \end{pmatrix} = r_1 \cdot \underline{v}_1 + r_2 \underline{v}_2 = r_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} + r_2 \cdot \begin{pmatrix} 3 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} r_1 + 3r_2 \\ 2r_1 - r_2 \end{pmatrix} = \begin{pmatrix} 5 \\ 2 \end{pmatrix} \qquad \begin{aligned} r_1 + 3r_2 &= 5 \\ 2r_1 - r_2 &= 2 \end{aligned}$$

$$\left(\begin{array}{cc|c} 1 & 3 & 5 \\ 2 & -1 & 2 \end{array}\right) \xrightarrow{\cdot(-2)} \left(\begin{array}{cc|c} 1 & 3 & 5 \\ 0 & -7 & -8 \end{array}\right) \qquad \begin{aligned} r_1 + 3r_2 &= 5 \\ -7r_2 &= -8 \end{aligned}$$

$$\begin{pmatrix} 5 \\ 2 \end{pmatrix} = \frac{11}{7} \cdot \underline{v}_1 + \frac{8}{7} \cdot \underline{v}_2 \qquad \begin{aligned} r_1 &= 5 - 3 \cdot (8/7) = 11/7 \\ r_2 &= 8/7 \end{aligned}$$

In general: $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$\begin{aligned} r_1 + 3r_2 &= v_1 \\ 2r_1 - r_2 &= v_2 \end{aligned} \qquad \left(\begin{array}{cc|c} 1 & 3 & v_1 \\ 2 & -1 & v_2 \end{array}\right) \xrightarrow{\cdot(-2)} \left(\begin{array}{cc|c} 1 & 3 & v_1 \\ 0 & -7 & v_2 - 2v_1 \end{array}\right)$$

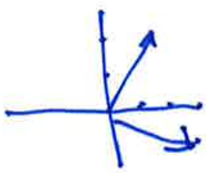
one solution

For any $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$, we have that $\underline{v} = r_1 \cdot \underline{v}_1 + r_2 \underline{v}_2$

Conclusion: $\text{span}(\underline{v}_1, \underline{v}_2) = \mathbb{R}^2 =$ set of all 2-vectors

$\underline{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\underline{v}_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$

2-dim.



Ex: $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \quad \underline{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} \quad \underline{v}_3 = \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$

$V = \text{span}(\underline{v}_1, \underline{v}_2, \underline{v}_3)$ (V is called a vector space)

compute V.

$$r_1 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + r_2 \cdot \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + r_3 \cdot \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \underline{v}$$

$$\begin{pmatrix} r_1 + r_2 + r_3 \\ r_1 + 2r_2 + 4r_3 \\ r_1 - 2r_3 \end{pmatrix} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & v_1 \\ 1 & 2 & 4 & v_2 \\ 1 & 0 & -2 & v_3 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & v_1 \\ 0 & 1 & 3 & v_2 - v_1 \\ 0 & -1 & -3 & v_3 - v_1 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & v_1 \\ 0 & 1 & 3 & v_2 - v_1 \\ 0 & 0 & 0 & v_2 + v_3 - 2v_1 \end{array}\right)$$

$v_2 + v_3 - 2v_1 \neq 0$: no solutions, i.e. $\underline{v}$ is not in $\text{span}(\underline{v}_1, \underline{v}_2, \underline{v}_3)$

$v_2 + v_3 - 2v_1 = 0$: inf. many solutions, i.e. $\underline{v}$ is in $\text{span}(\underline{v}_1, \underline{v}_2, \underline{v}_3)$

$v_3 = 2v_1 - v_2$

$\Downarrow$

<u>Conclusion:</u> $\text{span}(\underline{v}_1, \underline{v}_2, \underline{v}_3) = \left\{ \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} : v_2 + v_3 - 2v_1 = 0 \right\} = \text{span}(\underline{u}_1, \underline{u}_2)$ <u>2-dim.</u>

$$\begin{pmatrix} v_1 \\ v_2 \\ 2v_1 - v_2 \end{pmatrix} = \begin{pmatrix} v_1 \\ 0 \\ 2v_1 \end{pmatrix} + \begin{pmatrix} 0 \\ v_2 \\ -v_2 \end{pmatrix} = v_1 \cdot \underbrace{\begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}}_{=\,\underline{u}_1} + v_2 \underbrace{\begin{pmatrix} 0 \\ +1 \\ -1 \end{pmatrix}}_{=\,\underline{u}_2}$$

Homogeneous linear systems:

$$\left.\begin{matrix} a_{11}x_1 + \cdots + a_{1n}x_n = 0 \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n = 0 \end{matrix}\right\} \iff A \cdot \underline{x} = \underline{0}$$

$\underline{x} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ is called the <u>trivial solution</u>.

- one solution: $\underline{x} = \underline{0}$ (no non-trivial solutions)
- infinitely many solutions: at least one free variable (many non-trivial solutions)

<u>Ex</u>:

$$\left(\begin{matrix} -1 & 3 & 1 & 2 \\ 2 & 0 & 2 & -1 \\ -4 & 9 & 1 & 7 \end{matrix}\left|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right.\right) \begin{matrix} \\ 2 \\ -4 \end{matrix}$$

there will be free variables (at least one)

$M_{123,123} = -2 \cdot (-6) - 2 \cdot 3 = 6 \neq 0 \Rightarrow$ rk $= 3$ <u>one free variable x_4</u>

$$\left(\begin{matrix} -1 & 3 & 1 & 2 \\ 0 & 6 & 4 & 3 \\ 0 & -3 & -3 & -1 \end{matrix}\left|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right.\right) \to \left(\begin{matrix} -1 & 3 & 1 & 2 \\ 0 & -3 & -3 & -1 \\ 0 & 6 & 4 & 3 \end{matrix}\left|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right.\right) \begin{matrix} \\ \\ 2 \end{matrix}$$

$$\to \left(\begin{matrix} -1 & 3 & 1 & 2 \\ 0 & -3 & -3 & -1 \\ 0 & 0 & -2 & 1 \end{matrix}\left|\begin{matrix} 0 \\ 0 \\ 0 \end{matrix}\right.\right)$$

$-3x_2 = x_4 + 3x_3$

$-2x_3 = -x_4 \Rightarrow$

$\underline{x_3 = \frac{1}{2}x_4}$

$-3x_2 = x_4 + 3 \cdot (\frac{1}{2}x_4)$

$= \frac{5}{2}x_4$

$\underline{x_2 = -\frac{5}{6}x_4}$

$-x_1 = -3x_2 - x_3 - 2x_4$

$= -3 \cdot (-\frac{5}{6}x_4) - \frac{1}{2}x_4 - 2x_4$

$= \frac{5}{2}x_4 - \frac{1}{2}x_4 - 2x_4 = 0$

$\underline{x_1 = 0}$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \begin{pmatrix} 0 \\ -5/6\, x_4 \\ 1/2 \cdot x_4 \\ x_4 \end{pmatrix}, \quad x_4 \text{ free}$$

$$= x_4 \cdot \begin{pmatrix} 0 \\ -5/6 \\ 1/2 \\ 1 \end{pmatrix} = \frac{1}{6} \cdot x_4 \begin{pmatrix} 0 \\ -5 \\ 3 \\ 6 \end{pmatrix}$$

Conclusion:

The solution of the linear system is the span

$$\text{span}(\underline{w}), \quad \underline{w} = \begin{pmatrix} 0 \\ -5 \\ 3 \\ 6 \end{pmatrix}$$

In general:

Null(A) = the set of all solutions of $\underbrace{A \cdot \underline{x} = \underline{0}}_{\text{homogeneous linear system}}$
(null space of A)

$$= \text{span}(\underline{w}_1, \underline{w}_2, \ldots, \underline{w}_r)$$

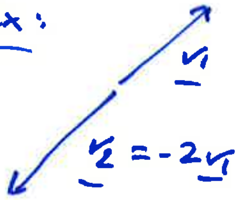
where $r = \#$ free variables $= n - \text{rk}(A)$

2 Linear independence of vectors

$\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$: vectors of the same size

Defn:
The set $\{\underline{v}_1, \ldots, \underline{v}_n\}$ of vectors is <u>linearly dependent</u> if at least one of the vectors is a linear combination of the others, and otherwise they are <u>linearly independent</u>.

Ex:

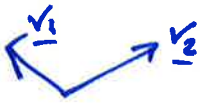


$\text{span}(\underline{v}_1, \underline{v}_2) = \text{span}(\underline{v}_1)$

$\underline{v}_1, \underline{v}_2$ are <u>linearly dependent</u> : $\underline{v}_2 = -2\underline{v}_1$ linear dependency relation

$\underline{0} = -2\underline{v}_1 - \underline{v}_2$

$\Updownarrow$



$\{\underline{v}_1, \underline{v}_2\}$ are <u>linearly independent</u>

Note: Any linear dependency relation can be written
$c_1\underline{v}_1 + \ldots + c_r\underline{v}_r = \underline{0}$
with at least one $c_i \neq 0$

Method to determine if $\{\underline{v}_1, \ldots, \underline{v}_n\}$ are linearly independent:

Consider the eqn: $\boxed{c_1 \cdot \underline{v}_1 + c_2 \cdot \underline{v}_2 + \ldots + c_n \underline{v}_n = \underline{0}}$

$\{\underline{v}_1, \ldots, \underline{v}_n\}$ <u>linearly independent</u>

One solution: $\underline{x} = \underline{0}$
(only trivial solution)
(no lin. dependency relation)

$\{\underline{v}_1, \ldots, \underline{v}_n\}$ <u>linearly dependent</u>

Infinitely many solutions
(non-trivial solution)

$$c_1\underline{v}_1 + c_2\underline{v}_2 + \ldots + c_n \cdot \underline{v}_n = \underline{0} \iff \underbrace{\begin{pmatrix} \underline{v}_1 & \underline{v}_2 & \ldots & \underline{v}_n \end{pmatrix}}_{A} \cdot \underbrace{\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{pmatrix}}_{\underline{x}} = \underbrace{\begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}}_{\underline{0}}$$

Method:

$\{\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n\} \rightsquigarrow A = (\underline{v}_1 | \underline{v}_2 | \cdots | \underline{v}_n)$: $A \cdot \underline{x} = \underline{0}$

i) Gauss: $(\underline{v}_1 | \underline{v}_2 | \cdots | \underline{v}_n) \to$ find pivots

(pivot in all col's) $\mathrm{rk}(A) = n \Leftrightarrow$ linearly independent

$\mathrm{rk}(A) < n \Leftrightarrow$ linearly dependent

ii) Minors: compute rk A using minors

Special case: A is a square matrix

$|A| \neq 0 \Leftrightarrow \mathrm{rk}\, A = n \Leftrightarrow$ linearly independent

$|A| = 0 \Leftrightarrow \mathrm{rk}\, A < n \Leftrightarrow$ linearly dependent

The case $\mathrm{rk}(A) < n$:

There are typically pivot positions in some columns, but not all.

The vectors that correspond to pivot columns are linearly independent, and form a maximal set of linearly independent vectors among $\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \ldots, \underline{v}_n\}$

Ex: $\underline{v}_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix}1\\2\\0\end{pmatrix}$ $v_3 = \begin{pmatrix}1\\4\\-2\end{pmatrix}$ $\rightsquigarrow A = \begin{pmatrix}1&1&1\\1&2&4\\1&0&-2\end{pmatrix}$

$$\left(\begin{array}{ccc|c}1&1&1&0\\1&2&4&0\\1&0&-2&0\end{array}\right) \begin{matrix}\\ \leftarrow -1 \\ \leftarrow -1\end{matrix} \to \left(\begin{array}{ccc|c}1&1&1&0\\0&1&3&0\\0&-1&-3&0\end{array}\right) \begin{matrix}\\ \\ \leftarrow 1\end{matrix} \to \left(\begin{array}{ccc|c}1&1&1&0\\0&1&3&0\\0&0&0&0\end{array}\right)$$

$\text{Rk}\,A = 2 < n = 3$: $\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ linearly dependent

Dependency relation:

$3c_3 - c_3 = \underline{2c_3}$

$$\left(\begin{array}{ccc|c}1&1&1&0\\0&1&3&0\\0&0&0&0\end{array}\right)$$

$c_1 + c_2 + c_3 = 0 \Rightarrow c_1 = -c_2 - c_3$

$c_2 + 3c_3 = 0 \Rightarrow c_2 = \underline{-3c_3}$

c_1, c_2 : basic

c_3 : free

$$\Downarrow$$

$$\begin{pmatrix}c_1\\c_2\\c_3\end{pmatrix} = \begin{pmatrix}2c_3\\-3c_3\\c_3\end{pmatrix} = c_3 \cdot \begin{pmatrix}2\\-3\\1\end{pmatrix}$$

infinitely many solutions

$\underline{c_3 = 1}$ gives one: $c_1 = 2 \quad c_2 = -3 \quad c_3 = 1$

$$2\underline{v}_1 - 3\underline{v}_2 + 1 \cdot \underline{v}_3 = \underline{0}$$

Since c_3 is free, can express v_3 as a linear comb. of $\underline{v}_1, \underline{v}_2$ $\longrightarrow$

$$\boxed{\underline{v}_3 = -2\underline{v}_1 + 3\underline{v}_2}$$

linear dependency relation

$$\Downarrow$$

$\{\underline{v}_1, \underline{v}_2\}$ linearly independent

$\text{span}(\underline{v}_1, \underline{v}_2, \underline{v}_3) = \text{span}(\underline{v}_1, \underline{v}_2)$

3 Rank and linear independence

$\{\underline{v}_1, \underline{v}_2, \dots, \underline{v}_n\} \longleftrightarrow A = (\underline{v}_1 | \underline{v}_2 | \dots | \underline{v}_n)$

$\mathrm{Rk}(A) =$ the maximal number of linearly independent vectors among $\{\underline{v}_1, \dots, \underline{v}_n\}$

Ex: $\underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix}$ $\underline{v}_3 = \begin{pmatrix} 1 \\ 3 \\ -1 \end{pmatrix}$ $\underline{v}_4 = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$\{\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4\}$ lin. dependent

$$A = \begin{pmatrix} \boxed{\begin{matrix} 1 & 1 & 1 \\ 1 & -2 & 3 \\ 1 & 3 & -1 \end{matrix}} & \begin{matrix} 1 \\ -1 \\ 0 \end{pmatrix} \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 2 & -2 \\ 0 & 2 & -2 & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & -3 & 2 & -2 \\ 0 & 0 & -2/3 & -7/3 \end{pmatrix}$$

$\underline{\mathrm{Rk}(A) = 3}$

$\{\underline{v}_1, \underline{v}_2, \underline{v}_3\}$ linearly independent

$\mathrm{span}(\underline{v}_1, \underline{v}_2, \underline{v}_3, \underline{v}_4) = \mathrm{span}(\underline{v}_1, \underline{v}_2, \underline{v}_3)$

$v_4 = c_1 \cdot \underline{v}_1 + c_2 \cdot \underline{v}_2 + c_3 \underline{v}_3$

$\boxed{\textbf{Note:}\ \mathrm{Rk}(A) = \dim \mathrm{span}(\underline{v}_1, \dots, \underline{v}_n)}$