Plan:

1. Extended Review: Lecture 3
2. Eigenvalues and eigenvectors
3. Diagonalization

Reminder: Plenary Session I on Mond. at 1700-2000 in A1-040

Problems: Selected from Lecture 1-4 (Key problems + Workbook problems)

1. Review: Lecture 3

$\{\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}\}$ ⟶ $A = (\underline{v_1} | \underline{v_2} | \dots | \underline{v_n})$

m-vectors $\qquad$ m×n-matrix

Key notions:

- linear combination of $\{\underline{v_1} \dots \underline{v_n}\}$
- span $(\underline{v_1}, \dots, \underline{v_n})$
- linear independence of $\{\underline{v_1} \dots \underline{v_n}\}$

V vector space: $\begin{cases} \underline{v}, \underline{w} \text{ in } V \Rightarrow \\ \underline{v}+\underline{w} \text{ in } V,\ r\underline{v} \text{ in } V \end{cases}$

$V = \text{span}(\underline{v_1}, \dots, \underline{v_n})$ vector space

$\boxed{\dim V = rk(A)}$ ← # pivot pos. in A

$\{\underline{v_1}, \dots \underline{v_n}\}$ linearly independent

⇕

no vector is in the span of the others

⇕

$x_1 \underline{v_1} + \dots + x_n \underline{v_n} = \underline{0}$ has only the trivial solution ⟷ $A \cdot \underline{x} = \underline{0}$ has only the zero (trivial) solution

⇕

$rk(A) = n$ ⟵ no free variables

$V = \text{span}(\underline{v}_1, \dots, \underline{v}_n)$:

A subset $B \subseteq \{\underline{v}_1, \dots, \underline{v}_n\}$ is called a base for V if

i) $\text{span}(B) = V$
ii) B is linearly independent

You may choose B to be the vectors corresponding to pivot positions in A. $\longrightarrow$ The number of vectors in B will be $\dim(V)$.

- Null(A): $\text{Null}(A)$ = Solutions of $A\underline{x} = \underline{0}$

$W = \text{Null}(A)$ is a vector space

$\boxed{\dim W = n - \text{rk}(A)}$

$W = \text{span}(\underline{w}_1, \dots, \underline{w}_r)$

$r = \dim W$ if the vectors $\underline{w}_1, \dots, \underline{w}_r$ are chosen minimally

Important fact:

$V = \text{span}(\underline{v}_1 \dots \underline{v}_n)$
$W = \text{Null}(A)$

$$\boxed{\begin{aligned} \dim V &= \text{rk}(A) \\ \dim W &= n - \text{rk}(A) \end{aligned}}$$

$\dim V = \text{rk}(A) \longleftarrow$ # pivot pos. in A

$\dim W = n - \text{rk}(A) \longleftarrow$ # col's without pivots in A

Ex: $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 0 \\ 3 & 4 & 2 \end{pmatrix} \leadsto \underline{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix} \quad \underline{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \quad \underline{v}_3 = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix}$

$\det A = 1 \cdot (-2) + 2 \cdot 1 = 0$
$\text{rk}\, A = 2$

$V = \text{span}(\underline{v}_1, \underline{v}_2, \underline{v}_3)$
$\dim V = 2$
$B = \{\underline{v}_1, \underline{v}_2\}$ base of $V \longleftarrow \{\underline{v}_1, \underline{v}_2\}$ lin. independent

$\underline{v}_3$ is in $\text{span}(\underline{v}_1, \underline{v}_2)$

$W = \text{Null}(A)$: Sol. of $A \cdot \underline{x} = \underline{0}$
$\dim W = 1$ (one free var. x_3)
$\underline{w}_1 = \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix} \quad W = \text{span}(\underline{w}_1)$

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 0 \\ 3 & 4 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ \cancel{0} & \cancel{1} & \cancel{-1} & \cancel{0} \end{array}\right)$$

$x_1 = -x_2 - x_3$
$x_2 = x_3$

$$\underline{x} = \begin{pmatrix} -2x_3 \\ x_3 \\ x_3 \end{pmatrix} = x_3 \cdot \begin{pmatrix} -2 \\ 1 \\ 1 \end{pmatrix}$$

$-2\underline{v}_1 + \underline{v}_2 + \underline{v}_3 = \underline{0}$

$\boxed{\underline{v}_3 = 2\underline{v}_1 - \underline{v}_2}$

② Eigenvectors and eigenvalues

Linear transformation

$f(\underline{x}) = A \cdot \underline{x}$

A ($n \times n$-matrix)

Ex: $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

eigen-vector

$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \\ 4 \end{pmatrix} = 4 \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ eigen-value

$\begin{pmatrix} 2 \\ 1 \end{pmatrix} \mapsto \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ 5 \end{pmatrix} \neq \lambda \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Defn: A ~~number~~ number λ is called an eigenvalue of A if $\boxed{A \cdot \underline{x} = \lambda \cdot \underline{x}}$ has non-trivial solutions for $\underline{x}$. If λ is an eigenvalue of A, then all solutions of $A \cdot \underline{x} = \lambda \underline{x}$ are called eigenvectors for A with eigenvalue λ.

Ex: $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$: $\begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \cdot \begin{pmatrix} x \\ y \end{pmatrix}$

$$3x + y = \lambda x$$
$$x + 3y = \lambda y$$

$$(3x - \lambda x) + y = 0$$
$$x + (3y - \lambda y) = 0$$

$$\begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix} \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

non-trivial solutions

$$\begin{vmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{vmatrix} = 0$$

$$(3-\lambda)^2 - 1 = 0$$
$$9 - 6\lambda + \lambda^2 - 1 = 0$$
$$\lambda^2 - 6\lambda + 8 = 0$$

characteristic eqn.

$\lambda_1 = \underline{4}$, $\lambda_2 = \underline{2}$ ← eigenvalues

In general:

$$A \cdot \underline{x} = \lambda \underline{x}$$
$$A \cdot \underline{x} - \lambda \underline{x} = \underline{0}$$
$$A \cdot \underline{x} - \lambda I \cdot \underline{x} = \underline{0}$$
$$\boxed{(A - \lambda I)\underline{x} = \underline{0}}$$

homogeneous linear system

Method for finding eigenvalues:

$\underset{n\times n}{A} \longrightarrow$ Characteristic equation:

$$|A-\lambda I| = 0$$

$(A-\lambda I)\underline{x}=\underline{0}$ has non-trivial solutions

Solutions for λ = eigenvalues of A

Ex: $A=\begin{pmatrix}3&1\\1&3\end{pmatrix}$: $\begin{vmatrix}3-\lambda & 1\\ 1 & 3-\lambda\end{vmatrix}=0$

$(3-\lambda)^2-1=0$

$\lambda^2-6\lambda+8=0$

$\lambda_1=4\ ,\ \lambda_2=2$

Ex: $A=\begin{pmatrix}a&b\\c&d\end{pmatrix}$: $\begin{vmatrix}a-\lambda & b\\ c & d-\lambda\end{vmatrix}=0$

$(a-\lambda)(d-\lambda)-bc=0$

$\lambda^2-(a+d)\lambda+(ad-bc)=0$

$$\boxed{\lambda^2-\mathrm{tr}(A)\cdot\lambda+\det(A)=0}$$

Formula for characteristic eqn. for all 2×2-matrices.

$\mathrm{tr}\,A$ = sum of all diagonal entries $= a_{11}+a_{22}+\dots+a_{nn}$

Method for finding eigenvectors when the eigenvalues are known.

Ex: $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

Eigenvalues: $\lambda_1 = 4$, $\lambda_2 = 2$

$\lambda = 4$: $A\underline{x} = \lambda \underline{x}$

$(A - \lambda I)\underline{x} = \underline{0} \xrightarrow[\lambda=4]{} \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\uparrow$

$\begin{pmatrix} 3-\lambda & 1 \\ 1 & 3-\lambda \end{pmatrix}$

$-1x + 1 \cdot y = 0$

$\begin{cases} x = y \\ y = \text{free} \end{cases}$

$\underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

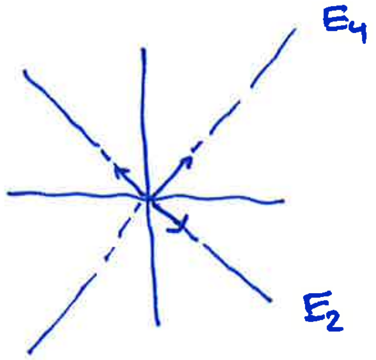
space of all eigenvectors for $\lambda = 4$

$E_4 = \text{Null}(A - 4I)$

$= \text{span}(\underline{w_1})$, $\underline{w_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = 2$: $\left(\begin{array}{cc|c} 1 & 1 & 0 \\ \cancel{1} & \cancel{1} & \cancel{0} \end{array}\right)$ $\quad x + y = 0$, y free $\quad \underline{x} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$E_2 = \text{span}(\underline{w_2})$, $\underline{w_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$



$A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

<u>In general:</u> $E_\lambda = \text{Null}(A-\lambda I)$

= all eigenvectors for A with eigenvalue λ

$(A-\lambda I)\cdot \underline{x} = \underline{0}$ ← solve this homogeneous linear system

$\dim E_\lambda = \#$ free variables

<u>Facts:</u>

A $(n \times n)$

* The characteristic equation has the form

$$|A-\lambda I| = 0 \Leftrightarrow \underbrace{(-\lambda)^n + \ldots}_{\text{lower degree terms}} = 0$$

It is a polynomial equation of degree $\underline{n}$.

* If A is <u>symmetric</u>, then A has n eigenvalues $\lambda_1, \ldots, \lambda_n$ (counted with multiplicity)

* If A has n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (counted with multiplicity), then

$$|A| = \lambda_1 \cdot \lambda_2 \cdot \ldots \cdot \lambda_n$$
$$\text{tr}(A) = \lambda_1 + \lambda_2 + \ldots + \lambda_n$$

* For an eigenvalue λ of multiplicity m, we have that

$$1 \leq \dim E_\lambda \leq m$$

\# free var. $(A-\lambda I)\underline{x} = \underline{0}$

If A is symmetric, then $\dim E_\lambda = m$ for all λ.

Ex: $A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$: $\begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = 0$

$\lambda^2 + 1 = 0$

$\lambda^2 = -1$ ← no. eigenvalues

$A = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$: $\begin{vmatrix} 1-\lambda & 3 \\ 0 & 1-\lambda \end{vmatrix} = 0$

$(1-\lambda)^2 = 0$

$\lambda^2 - 2\lambda + 1 = 0$

$\lambda = \frac{2 \pm \sqrt{4 - 4\cdot 1}}{2}$

$= \frac{2 \pm 0}{2}$

$\lambda_1 = 1 \quad \lambda_2 = 1$

$\lambda = 1$ has multiplicity 2 → $\lambda_1 = \lambda_2 = 1$

$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix}$: $\begin{vmatrix} 1-\lambda & 0 & 3 \\ 0 & -2-\lambda & 0 \\ 3 & 0 & 1-\lambda \end{vmatrix} = 0$

$(-2-\lambda)\cdot\left((1-\lambda)^2 - 9\right) = 0$

$(-2-\lambda)\cdot(\lambda^2 - 2\lambda - 8) = 0$

$\lambda = -2$ or $\lambda^2 - 2\lambda - 8 = 0$

$\lambda = \frac{2 \pm \sqrt{4 - 4\cdot(-8)}}{2}$

$= \frac{2 \pm 6}{2}$

$\lambda = -2, \lambda = 4$

$\underbrace{\lambda_1 = -2, \lambda_2 = -2}_{\text{mult. 2}}, \lambda_3 = 4$

3 Diagonalization

A ($n \times n$ matrix)

Defn: A is diagonalizable if there is a diagonal matrix D and an invertible matrix P such that

$$P^{-1}AP = D$$

Key result:

A is diagonalizable if and only if:

$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix} \rightarrow$ i) A has n eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ (counted with multiplicity), and

$P = \left(\underline{v_1} \,|\, \underline{v_2} \,|\, \ldots \,|\, \underline{v_n} \right) \rightarrow$ ii) A has n linearly independent eigenvectors $\underline{v_1}, \underline{v_2}, \ldots, \underline{v_n}$ (with $A\underline{v_i} = \lambda_i \underline{v_i}$)

In particular:

A symmetric $\Rightarrow$ A diagonalizable

Why?

$$A \cdot P = A \cdot \left(\underline{v_1} | \underline{v_2} | \ldots | \underline{v_n} \right) = \left(A\underline{v_1} | A\underline{v_2} | \ldots | A\underline{v_n} \right)$$

$$= \left(\lambda_1 \underline{v_1} | \lambda_2 \underline{v_2} | \ldots | \lambda_n \underline{v_n} \right) = \left(\underline{v_1} | \underline{v_2} | \ldots | \underline{v_n} \right) \cdot \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & \ddots & \lambda_n \end{pmatrix}$$

$$= P \cdot D$$

$$AP = PD \Rightarrow \boxed{P^{-1}AP = D}$$

Ex: $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$

$\lambda_1 = 4, \quad \lambda_2 = 2$

$E_4 = \text{span}(\underline{w_1}), \quad \underline{w_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$E_2 = \text{span}(\underline{w_2}), \quad \underline{w_2} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$D = \begin{pmatrix} 4 & 0 \\ 0 & 2 \end{pmatrix}$

$P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$

$\boxed{P^{-1}AP = D}$

A diagonalizable

Result: (assume that A has n eigenvalues $\lambda_1, \dots, \lambda_n$)

> There are n linearly independent eigenvectors
>
> $\Updownarrow$
>
> For each eigenvalue λ (of multiplicity m), we have $\dim E_\lambda = m$.

Ex: $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

Eigenvalues:

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 0 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{vmatrix} = (1-\lambda)(1-\lambda)(3-\lambda) = 0$$

$\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 3$

For any upper triangular matrix, the eigenvalues are the diagonal entries.

$\underline{\lambda = 3:}$ $\left(\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$

$x = 0$
$y = 0$

$\underline{x} = \begin{pmatrix} 0 \\ 0 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$E_3 = \text{span}(\underline{w_3}), \quad \underline{w_3} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

$\underline{\lambda = 1:}$ $\left(\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right)$ $\quad y = 0$, $z = 0$, x free

$\underline{x} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = x \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$E_1 = \text{span}(\underline{w_1}), \quad \underline{w_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$P = \begin{pmatrix} 1 & ? & 0 \\ 0 & ? & 0 \\ 0 & ? & 1 \end{pmatrix}$

A is not diagonalizable

not enough eigenvectors

Consequence:

Result:

If A has n eigenvalues, all of multiplicity 1, then A is diagonalizable.

Note: ① A symmetric $\Rightarrow$ A diagonalizable

– but there are non-symmetric matrices that are diagonalizable as well

Ex: $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ not symmetric, not diagonalizable (no eigenvalues)

$A = \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix}$ not symmetric, diagonalizable $\lambda_1 = 1$, $\lambda_2 = 3$

$\underline{w}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ $\underline{w}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

② A has n eigenvalues of multiplicity 1 $\Rightarrow$ A diagonalizable

– but there are matrices with eigenvalues of higher multiplicity that are diagonalizable

Ex: $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ $\lambda = 1$ (mult. 2) A is not diagonalizable

$A = \begin{pmatrix} 1 & 0 & 3 \\ 0 & -2 & 0 \\ 3 & 0 & 1 \end{pmatrix}$ $\lambda = -2$ (mult 2) A is diagonalizable (it is symmetric)