Plan:

Review Lecture 7-8

1. Envelope theorems in the unconstrained case
2. Envelope theorems in the constrained case

Note:

* Next week: Differential eqn.
  Look at Appendix [DE], do the problems (A1–A10, Solutions on web page)
* Midterm exam
* Bordered Hessians is not in the curriculum this year.

Review: Constraind optimization (Lagrange Kuhn-Tucker)

Method:

1. Find ordinary candidate pts: Solve FOC+C / FOC+C+CSC
   Compute $f(\underline{x})$ for each cand. pt. ⤳ Find best candidate
2. Determine if best candidate is max/min.
   (a) Use SOC on best candidate $(\underline{x}^*; \underline{\lambda}^*)$
   $h(\underline{x}) = \mathcal{L}(\underline{x}; \underline{\lambda}^*)$: $h$ concave $\Rightarrow$ $\underline{x}^*$ is max
   $h$ convex $\Rightarrow$ $\underline{x}^*$ is min
   (b) Use EVT: if the set D of adm. pts is bounded, then there is a max/min.
   all pts that satisfy all constraints
   ⇓
   it must be one of these pts:
   i) all regular cand. pts (from 1)
   ii) all adm. pts where NDCQ fails

# EXAM RESULT

Midterm exam in GRA6035 Mathematics 12/10/2018

## Summary

| | |
|---|---|
| Grade A-B | 45.1% |
| Grade F | 4.0% |
| Average score | 13.7p (C) |

## Comments

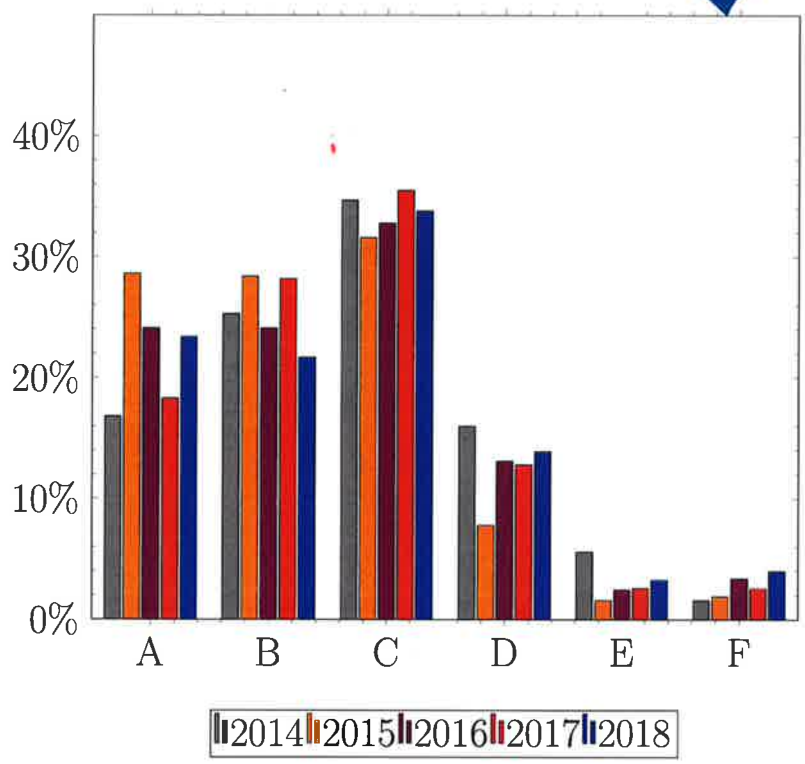
The results on the midterm exam were **overall very good**. Question 2,3,5,8 had the highest rate of wrong answers, and Question 6,8 had the lowest answering rate. This was to a large extent expected, as these were among the most difficult questions. Many questions were **identical to or very similar to Key Problems**. A summary of scores is given in the table below.

**Average score per problem**

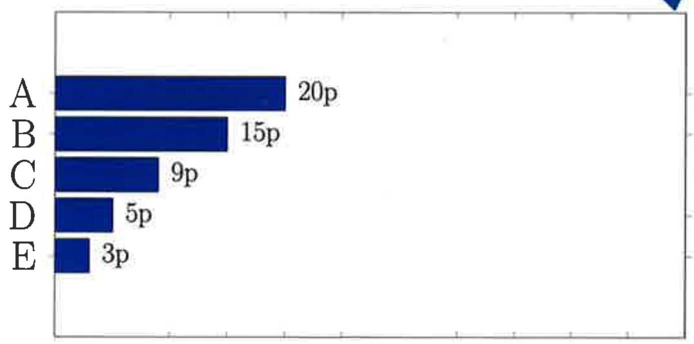
| | Correct A. | Wrong A. | Unansw. |
|---|---|---|---|
| Score: | 3p | -1p | 0p |
| Question 1: | 92% | 9% | 1% |
| Question 2: | 57% | 26% | 17% |
| Question 3: | 57% | 34% | 9% |
| Question 4: | 88% | 9% | 2% |
| Question 5: | 59% | 32% | 10% |
| Question 6: | 50% | 16% | 35% |
| Question 7: | 81% | 16% | 3% |
| Question 8: | 31% | 29% | 40% |

I expected better results on Question 2 and 5, and a higher answering rate on Question 6, and recommended that you **review all problems that you did not answer correctly**. In Q2, we must have $\mathbf{v}_2 = 3\mathbf{v}_1$ for the vectors to be linearly dependent, and this is not true for any $t$. In Q5, the matrix is diagonalizable since it is symmetric. Q6 is a standard Markov chain computation. Detailed solutions to all questions can be found at www.dr-eriksen.no.

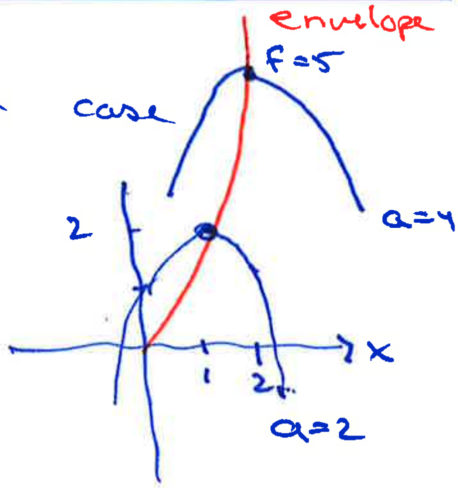
## Grade distribution last 5 years




## Grading scale



① Envelope theorems: Unconstrained case




Ex: $\max f(x) = 1+2x-x^2$

$f' = 2-2x = 0$

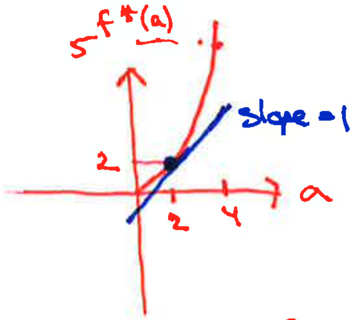
$x=1$

$f'' = -2 < 0$ : f concave

$x^* = 1$ is max pt

$f^* = f(1) = 2$ max value

Parametrized version of problem:




$\max f(x;a) = 1 + ax - x^2$

$f' = a - 2x = 0$

$x = a/2$

$f'' = -2 < 0$ f(x;a) concave fn.

$x^*(a) = a/2$ is max pt.

$f^*(a) = 1 + a \cdot a/2 - (a/2)^2$

$= 1 + a^2/2 - a^2/4$

$f^*(a) = 1 + a^2/4$ max value

$f^*(a) = 1 + a^2/4$ optimal value function

optimal (maximal) value function

Ex: $a=4$ $x^*(4) = 2$ $f^*(4) = 5$

$f^*(a)$ = max value for the given value of a

$$\frac{df^*(a)}{da} = \frac{2a}{4} = \frac{a}{2} \qquad \frac{df^*}{da}(2) = 1$$

**Envelope theorem: Unconstrained case**

If $x^*(a)$ is max/min in an unconstrained problem for each value of $a$. Then we have:

$$\frac{df^*(a)}{da} = \frac{\partial f(x;a)}{\partial a}\,(x = x^*(a))$$

**Ex:** $f(x;a) = 1 + ax - x^2 \rightsquigarrow$ $\max f(x;a)$, $f(x;a) = 1 + ax - x^2$

**Env. thm:**

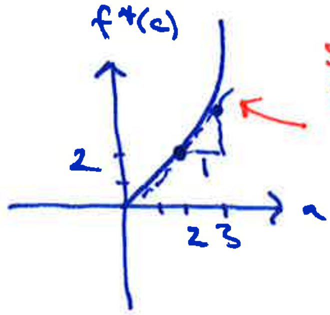
$$\frac{df^*(a)}{da} = \frac{\partial f}{\partial a}(x^*(a)) = x^*(a) = \underline{a/2}$$

$\frac{\partial f}{\partial a} = x$

$x^*(a) = a/2$

$\uparrow$ Slope of tangent of optimal value fn.

$\underline{a = 2}$: Max $x^*(2) = 1$, $f^*(2) = 2$




Slope of tangent line $= \underline{1}$

$$f^*(3) \simeq f^*(2) + (3-2)\cdot 1 = 2 + 1 = \underline{3}$$

estimate of new max value, following the tangent line

$f^*(a) = 1 + a^2/4$

$f^*(3) = 1 + 9/4 = \underline{3{,}25}$

**Ex:** Kelly Criterion

Even money bets with positive expected return.

$X_0$ : start capital

$B_1$ : bet

$X_1$ : capital after first bet

$X_0$ — **Win:** prob. $p > 1/2$, $X_1 = X_0 + B_1$

$X_0$ — **Lose:** prob. $q = 1 - p$, $X_1 = X_0 - B_1$

$$E(X_1 - X_0) = p \cdot B_1 + q \cdot (-B_1) = (p-q)B_1 > 0$$

**Ex:** $\left. \begin{array}{l} p = 0.6 \\ q = 0.4 \end{array} \right\}$ $E(X_1 - X_0) = 0.2 B_1$

**Assume:**

$n$ games, independent outcomes

$B_i = b \cdot X_{i-1}$ bet fixed prop. of capital each time

$0 \le b \le 1$

**After 1 game:**

**Win:** $X_1 = X_0 + bX_0 = X_0 \cdot (1+b)$

**Lose:** $X_1 = X_0 - bX_0 = X_0 \cdot (1-b)$

**After n games:**

$$X_n = X_0 \cdot (1+b)^W \cdot (1-b)^L$$

$W$ = # wins, $L$ = # losses, $W + L = n$

$$E\left[\tfrac{1}{n}\ln(X_n/X_0)\right] = E\left[\tfrac{1}{n}\ln\left((1+b)^W(1-b)^L\right)\right]$$

$$= E\left[\tfrac{1}{n}\cdot\left(W\ln(1+b) + L\cdot\ln(1-b)\right)\right]$$

$$\boxed{f(b) = p\cdot\ln(1+b) + q\cdot\ln(1-b)}$$

we maximize this fn.

$p$ parameter, $q = 1-p$

$$\tfrac{1}{n}\ln(X_n/X_0) = g \iff X_n = X_0 \cdot e^{gn}$$

$g$ : exponential rate of growth in capital per game

$\ln(X_n/X_0) = ng$

$X_n/X_0 = e^{gn}$ $\quad X_n = X_0 \cdot e^{gn}$

$$\max f(b) = p\cdot\ln(1+b) + q\cdot\ln(1-b), \quad 0 \le b \le 1$$

$$f'(b) = \frac{p}{1+b} + \frac{q}{1-b}\cdot(-1) = \frac{p(1-b) - q\cdot(1+b)}{(1+b)(1-b)}$$

$-(p+q)b = -b$ since $p+q=1$

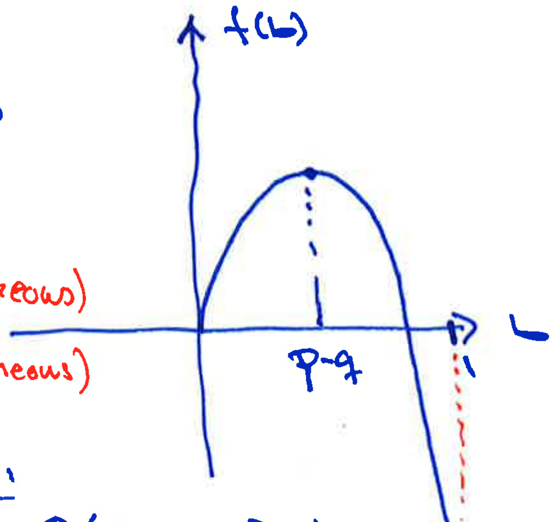
$$= \frac{p-q-pb-qb}{(1+b)(1-b)} = \frac{p-q-b}{(1+b)(1-b)} = 0$$

$$p-q-b=0 \iff b = \underline{p-q}$$

$$f''(b) = -\frac{p}{(1+b)^2} - \frac{(-q)}{(1-b)^2}\cdot(-1) = -\frac{p}{(1+b)^2} - \frac{q}{(1-b)^2} < 0$$

$f$ concave

$$b^*(p) = \underline{p-q} = p-(1-p) = 2p-1$$




$p$: decided outside the model (exogeneous)

$b$: decided in the model (endogeneous)

$$\frac{db^*(p)}{dp} = \underline{\underline{2}}$$

Ex:

$p = 0.60 \longrightarrow b = 0.60 - 0.40 = \underline{0.20}$

$p = 0.57 \qquad b = 0.57 - 0.43 = \underline{0.14}$

$b = p - q$ is called Kelly bet

## ② Envelope theorem: constrained case

**Ex:** max $f(x,y) = x + 3y$ when $x^2 + y^2 = 10$

what happens if 3 changes to 2?

what happens if $a$ is 11?


① **Solve** max $f = x + 3y$ when $x^2+y^2 = 10$ :
$x^2+y^2-10=0$

$$L = x + 3y - \lambda(x^2+y^2-10)$$

**FOC:** $L'_x = 1 - \lambda \cdot 2x = 0$ $\quad x = \frac{1}{2\lambda}$ $\quad (\lambda \neq 0)$

$L'_y = 3 - \lambda \cdot 2y = 0$ $\quad y = \frac{3}{2\lambda}$

**C:** $x^2+y^2-10=0$ $\quad \left(\frac{1}{2\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 - 10 = 0$

$$\frac{10}{(2\lambda)^2} = \frac{1+9}{(2\lambda)^2} = 10$$

$(2\lambda)^2 = 1 \quad 2\lambda = \pm 1 \quad \lambda = \pm 1/2$

**Cand pts:** $(x,y;\lambda) = \boxed{(1,3;1/2) \; f=10}$, $(-1,-3;-1/2)$ $f=-10$

best cand. for max.

**SOC:** $h(x,y) = x + 3y - \frac{1}{2}(x^2+y^2-10)$

$h'_x = 1 - x$ $\quad H(h) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$ $\quad D_1 = -1$, $D_2 = 1$ $\quad$ neg. defin. for all $x,y$

$h'_y = 3 - y$ $\quad$ $h$ concave

$\Downarrow$ SOC

$\boxed{x^* = 1 \quad y^* = 3 \quad \lambda^* = 1/2 \quad f^* = 10}$ $\quad$ $(x,y) = (1,3)$ is max

(2) What about: max $f = x + 3y$ wh $x^2 + y^2 = 11$

$x^2 + y^2 - 11 = 0$

Consider Lagrange problem with parameter $a$:

$$\max f(x,y;a) = x + 3y \quad \text{when } g(x,y;a) = x^2 + y^2 - a = 0$$

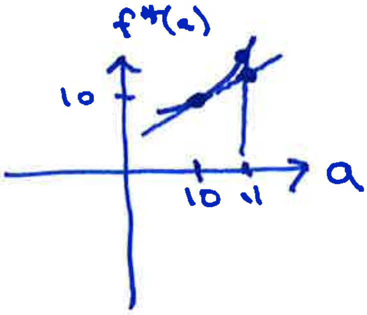
$a = 10$:

$x^*(10) = 1$

$y^*(10) = 3$

$\lambda^*(10) = 1/2$

$f^*(10) = 10$




$$L = x + 3y - \lambda \cdot (x^2 + y^2 - a) \qquad L = f(x,y;a) - \lambda \cdot g(x,y;a)$$

$$\frac{\partial L}{\partial a} = \lambda \quad \rightsquigarrow \quad \frac{\partial L}{\partial a}(x^*(a), y^*(a); \lambda^*(a)) = \lambda^*(a)$$

Env. thm: $\dfrac{df^*(a)}{da} = \dfrac{\partial L}{\partial a}(\underline{x}^*(a); \underline{\lambda}^*(a))$

$= \lambda^*(a)$

$a = 10$: $\dfrac{df^*(a)}{da} = \dfrac{1}{2}$

$$f^*(11) \cong \underbrace{f^*(10)}_{=10} + \underbrace{\Delta a}_{=11-10=1} \cdot \underbrace{\frac{df^*(a)}{da}}_{=1/2} = \underline{\underline{10.5}}$$

estimate of new max value

Envelope thm for Lagrange / Kuhn-Tucker problems

If $\underline{x}^*(a)$ is optimal (max/min), such that $(\underline{x}^*(a); \lambda^*(a))$ satisfies FOC+C / FOC+C+CSC, for all a, then we have:

$$\frac{df^*(a)}{da} = \frac{\partial L}{\partial a}(\underline{x}^*(a); \lambda^*(\underline{a}))$$

Note: max $f(x_1, \ldots, x_n)$ when $g(x_1, \ldots, x_n; a) = 0$

$$g(x_1, \ldots, x_n; a) = g(x_1, \ldots, x_n) - a = 0$$

$$L = f(x_1, \ldots, x_n) - \lambda \cdot (g(x_1, \ldots, x_n) - a)$$

$$\frac{\partial L}{\partial a} = -\lambda \cdot (-a)' = -\lambda \cdot (-1) = \underline{\lambda}$$

Interpretation of $\lambda$: $\lambda = \frac{df^*(a)}{da}$ when a is the constant in the constraint

to increase the constant in the constraint by 1, will mean that the max. value increases by approx. $\lambda$.

$\max \quad f(x,y;a) = x + ay \quad$ when $\quad g(x,y;a) = x^2 + y^2 - 10 = 0$

new Lagrange problem with parameter $a$:

$x^*(3) = 1$
$y^*(3) = 3$
$\lambda^*(3) = 1/2$
$f^*(3) = 10$

$L = x + ay - \lambda(x^2 + y^2 - 10)$

$\frac{\partial L}{\partial a} = y$

$\Downarrow$ Envelope thm:

$$\frac{df^*(a)}{da} = \frac{\partial L}{\partial a}(x^*(a), y^*(a); \lambda^*(a)) = y^*(a)$$

$\underline{a=3}$: $\frac{df^*(a)}{da} = y^*(3) = 3$

$$f^*(4) \simeq f^*(3) + \Delta a \cdot \frac{df^*(a)}{da} = 10 + 1 \cdot 3 = \underline{\underline{13}}$$

estimate of new max value