Selected problems:

(A) Key problems: 3.3, 4.1de, 4.3

(B) Workbook problems: 1.15, 2.18b, 3.12, 3.14, 4.3b, 4.5, 4.6, 4.9, 4.10

(C) Midterm exam 10/2017: Question 1-5

included in the notes after the lecture.

(A) Key problems:

3.3.

$$A = \begin{pmatrix} -1 & 3 & 1 & 2 & 4 \\ 2 & 0 & 2 & -1 & 3 \\ -4 & 9 & 1 & 7 & 9 \end{pmatrix}$$

Null(A) = Solutions of $A \cdot \underline{x} = \underline{0}$

$$\hat{A} = \left(\begin{array}{ccccc|c} -1 & 3 & 1 & 2 & 4 & 0 \\ 2 & 0 & 2 & -1 & 3 & 0 \\ -4 & 9 & 1 & 7 & 9 & 0 \end{array}\right) \begin{matrix} \\ \leftarrow 2 \\ \leftarrow -4 \end{matrix} \to \left(\begin{array}{ccccc|c} -1 & 3 & 1 & 2 & 4 & 0 \\ 0 & +6 & 4 & 3 & 11 & 0 \\ 0 & -3 & -3 & -1 & -7 & 0 \end{array}\right) \updownarrow$$

$$\left(\begin{array}{ccccc|c} -1 & 3 & 1 & 2 & 4 & 0 \\ 0 & -3 & -3 & -1 & -7 & 0 \\ 0 & 6 & 4 & 3 & 11 & 0 \end{array}\right) \begin{matrix} \\ \\ \leftarrow 2 \end{matrix} \to \left(\begin{array}{ccccc|c} -1 & 3 & 1 & 2 & 4 & 0 \\ 0 & -3 & -3 & -1 & -7 & 0 \\ 0 & 0 & -2 & 1 & -3 & 0 \end{array}\right)$$

echelon form
two free var's (x_4, x_5)

$$-2x_3 + x_4 - 3x_5 = 0 \Rightarrow x_3 = \frac{-x_4 + 3x_5}{-2} = \frac{1}{2}x_4 - \frac{3}{2}x_5$$

$$-3x_2 - 3x_3 - x_4 - 7x_5 = 0$$

$$\Rightarrow \frac{-3x_2}{-3} = \frac{3\left(\frac{1}{2}x_4 - \frac{3}{2}x_5\right) + x_4 + 7x_5}{-3} = \frac{\frac{5}{2}x_4 + \frac{5}{2}x_5}{-3}$$

$$x_2 = -\frac{5}{6}x_4 - \frac{5}{6}x_5$$

$$-x_1 + 3x_2 + x_3 + 2x_4 + 4x_5 = 0$$

$$\Rightarrow -x_1 = -3\left(-\frac{5}{6}x_4 - \frac{5}{6}x_5\right) - \left(\frac{1}{2}x_4 - \frac{3}{2}x_5\right) - 2x_4 - 4x_5$$

$$= \frac{5}{2}x_4 - \frac{1}{2}x_4 - 2x_4 + \frac{5}{2}x_5 + \frac{3}{2}x_5 - 4x_5 = 0 \Rightarrow x_1 = 0$$

$$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ -5/6\, x_4 \; -5/6\, x_5 \\ 1/2\, x_4 - 3/2\, x_5 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 0 \\ -5/6\, x_4 \\ 1/2\, x_4 \\ x_4 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -5/6\, x_5 \\ -3/2\, x_5 \\ 0 \\ x_5 \end{pmatrix}$$

$$= x_4 \cdot \underbrace{\begin{pmatrix} 0 \\ -5/6 \\ 1/2 \\ 1 \\ 0 \end{pmatrix}}_{\underline{w_1}} + x_5 \cdot \underbrace{\begin{pmatrix} 0 \\ -5/6 \\ -3/2 \\ 0 \\ 1 \end{pmatrix}}_{\underline{w_2}}$$

$\text{Null}(A) = \underline{\text{span}(\underline{w_1}, \underline{w_2})}$ $\quad \underline{r = 2}$

Base for Null(A): $\{\underline{w_1}, \underline{w_2}\}$

(lin. independence is automatic)

4.1 d) $A = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 4 \end{pmatrix}$

Eigenvalues: $\begin{vmatrix} 4-\lambda & 0 & 1 \\ 0 & 5-\lambda & 0 \\ 1 & 0 & 4-\lambda \end{vmatrix} = 0$

$$(5-\lambda)\cdot\left((4-\lambda)^2 - 1\right) = 0$$

$$(5-\lambda)\cdot(\lambda^2 - 8\lambda + 15) = 0$$

$\underline{\lambda_1 = 5}$ or $\lambda^2 - 8\lambda + 15 = 0$

$$\lambda = \frac{8 \pm \sqrt{64 - 4\cdot 15}}{2} = \frac{8 \pm 2}{2}$$

$$\underline{\lambda_2 = 5}, \quad \underline{\lambda_3 = 3}$$

Eigenvectors:

$\underline{\lambda = 5}$: E_5

$$\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix} \underline{x} = \underline{0} \longrightarrow \begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$-x \quad + z = 0 \Rightarrow x = z$ $\quad$ 2 free

y free

z free

Base: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ y \\ 0 \end{pmatrix} + \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} = \underline{y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}}$$

$\underline{\lambda=3:}$ $\quad \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \begin{aligned} x + z &= 0 \\ 2y &= 0 \end{aligned}$ $\quad \begin{aligned} x &= -z \\ y &= 0 \\ z &= z \text{ (free)} \end{aligned}$

E_3 1 free

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z \cdot \underline{\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

$\underline{\text{Base}}$: $\left\{ \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix} \right\}$

$E_3 = \text{span}(\underline{v_3})$, $\underline{v_3} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

4.1 c) $\quad A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\underline{(2-\lambda) \cdot [(2-\lambda)^2 - 1]} \underline{- 1 \cdot [2 - \lambda - 1]} \underline{+ 1 \cdot [1 - (2-\lambda)]} = 0$$

$$(2-\lambda) \cdot [(2-\lambda)^2 - 1] \quad \underbrace{-1 + \lambda - 1 + \lambda}_{2\lambda - 2 = \underline{2(\lambda-1)}} = 0$$

$$(2-\lambda) \cdot (\lambda^2 - 4\lambda + 3)$$

$$(2-\lambda) \cdot \underline{(\lambda-1)}(\lambda - 3) + 2\underline{(\lambda-1)} = 0$$

$$(\lambda-1) \cdot [(2-\lambda)(\lambda-3) + 2] = 0$$

$\underline{\lambda = 1}$ or $-\lambda^2 + 5\lambda - 4 = 0$

$$\lambda = \frac{-5 \pm \sqrt{25-16}}{-2} = \frac{-5 \pm 3}{-2}$$

$\underline{\lambda = 1}$ or $\underline{\lambda = 4}$

$\underline{E_1}$: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ $\quad \begin{aligned} x &= -y - z \\ y &= y \\ z &= z \end{aligned}$ $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \underline{y \cdot \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z \cdot \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$

$\underline{E_4}$: $\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}$ $\cdots\cdots \to$ $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

4.3: $A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

A symmetric
$\Downarrow$
A diagonalizable.

$$\begin{vmatrix} -\lambda & 1 & 1 & 0 \\ 1 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 1 \\ 0 & 1 & 1 & -\lambda \end{vmatrix} = 0$$

Notice:
rk $A = 2$, $|A| = 0$
$\lambda = 0$

$$-\lambda \cdot \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 1 & 0 \\ 0 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 1 & 0 \\ -\lambda & 0 & 1 \\ 1 & 1 & -\lambda \end{vmatrix}$$

$$(-\lambda)\left[-\lambda \cdot (\lambda^2 - 1) + 1 \cdot \lambda\right] - 1 \cdot (-\lambda)(-\lambda) + 1 \cdot (-\lambda) \cdot \lambda = 0$$

$$-\lambda(-\lambda^3 + \lambda + \lambda) - \lambda^2 - \lambda^2$$

$$\lambda^4 - 2\lambda^2 - \lambda^2 - \lambda^2 = 0$$

$$\lambda^4 - 4\lambda^2 = 0$$

$$\lambda^2(\lambda^2 - 4) = 0$$

$\underline{\lambda_1 = \lambda_2 = 0}$, $\underline{\lambda_3 = 2}$, $\underline{\lambda_4 = -2}$

$|A - \lambda I| = 0$
$\lambda = 0$ has multiplicity 2
$\lambda_1 = \lambda_2 = 0$, $\lambda_3 = \ldots$ $\lambda_4 = \ldots$

B. Workbook problems

1.15.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2q \\ 2 & -3 & 2 & 4q \\ 3 & -2 & p & q \end{array}\right) \begin{matrix} \\ -2 \\ -3 \end{matrix} \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2q \\ 0 & -5 & 0 & 0 \\ 0 & -5 & p-3 & -5q \end{array}\right) \begin{matrix} \\ \\ -1 \end{matrix}$$

$$\to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 2q \\ 0 & -5 & 0 & 0 \\ 0 & 0 & p-3 & -5q \end{array}\right)$$

$p \neq 3$: $\left.\begin{array}{l} \operatorname{rk} A = 3 \\ \operatorname{rk} \hat{A} = 3 \end{array}\right\}$ one solution

$p = 3$: $\to q \neq 0$: $\left.\begin{array}{l} \operatorname{rk} A = 2 \\ \operatorname{rk} \hat{A} = 3 \end{array}\right\}$ no solutions

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 2q \\ 0 & -5 & 0 & 0 \\ 0 & 0 & 0 & -5q \end{array}\right)$$

$q = 0$: $\left.\begin{array}{l} \operatorname{rk} A = 2 \\ \operatorname{rk} \hat{A} = 2 \end{array}\right\}$ infinitely many solutions

(z free)

2.18 b)

$$A = \begin{pmatrix} t+3 & 5 & 6 \\ -1 & t-3 & -6 \\ 1 & 1 & t+4 \end{pmatrix}$$

$$|A| = (t+3)\cdot\left[(t-3)(t+4)+6\right] + 1\cdot\left(5(t+4)-6\right) + 1\cdot\left(-30-6(t-3)\right)$$

$$= (t+3)\left((t-3)(t+4)+6\right) + \underbrace{5t-6t+14-30+18}_{-t+2\,=\,-(t-2)}$$

$$= (t+3)\cdot(t^2+t-6) - (t-2)$$

$$= (t+3)\,\underline{(t-2)}(t+3) - \underline{(t-2)}$$

$$= (t-2)\left((t+3)^2-1\right) = (t-2)(t^2+6t+8)$$

$$= (t-2)(t+2)(t+4) = 0 \iff t=2,\ t=-2,\ t=-4$$

$\underline{t \neq 2,-2,-4:}$ $\quad \mathrm{rk}\,A = \underline{3}$ $\quad$ (since $|A| \neq 0$)

$\underline{t=2:}$ $\begin{pmatrix} 5 & 5 & 6 \\ -1 & -1 & -6 \\ 1 & 1 & 6 \end{pmatrix}$ $\quad 5\cdot(-6)-(-6) \neq 0 \Rightarrow \mathrm{rk}\,A = \underline{2}$

$\underline{t=-2:}$ $\begin{pmatrix} 1 & 5 & 6 \\ -1 & -5 & -6 \\ 1 & 1 & 2 \end{pmatrix}$ $\quad -1+5 \neq 0 \Rightarrow \mathrm{rk}\,A = \underline{2}$

$\underline{t=-4:}$ $\begin{pmatrix} -1 & 5 & 6 \\ -1 & -7 & -6 \\ 1 & 1 & 0 \end{pmatrix}$ $\quad 7+5 \neq 0 \Rightarrow \mathrm{rk}\,A = \underline{2}$

$$\boxed{\mathrm{rk}\,A = \begin{cases} 3, & t \neq 2,-2,4 \\ 2, & t = 2,-2,4 \end{cases}}$$

3.12. Note: $\underset{m\times n}{A} \rightsquigarrow \underbrace{A^T \cdot A}_{n\times n\text{-matrix}}$ $(n\times m)(m\times n)$

1)

Null(A): Solutions of $A\cdot \underline{x} = \underline{0}$

Null(A^TA): Solutions of $(A^TA)\underline{x} = \underline{0}$

a) $A\underline{x} = \underline{0} \Rightarrow A^T\cdot A\underline{x} = A^T\cdot \underline{0} = \underline{0} \Rightarrow (A^TA)\underline{x} = \underline{0}$ ← All vectors in Null(A) are in Null(A^TA).

b) $A^TA\underline{x} = 0 \Rightarrow \underline{x}^TA^TA\underline{x} = \underline{x}^T\cdot\underline{0} = \underline{0}$

$(A\underline{x})^T\cdot(A\underline{x}) = \underline{0} \Rightarrow A\underline{x} = 0$ ← All vectors in Null(A^TA) are in Null(A)

Note: for any vector $\underline{v}$, we have

$\underline{v}^T\cdot\underline{v} = (v_1 \dots v_n)\begin{pmatrix} v_1 \\ \vdots \\ v_n\end{pmatrix} = v_1^2 + v_2^2 + \dots + v_n^2$

so $\underline{v}^T\cdot\underline{v} = \underline{0} \Rightarrow \underline{v} = 0$

a) + b) : Null(A) = Null(A^TA).

2) $A\underline{x} = \underline{0}$ and $A^TA\underline{x} = \underline{0}$ has the same solutions (from 1) above), therefore the same degrees of freedom, and this means

$$\underbrace{n - rk(A)}_{\text{free var's in } A\underline{x}=\underline{0}} = \underbrace{n - rk(A^TA)}_{\text{free var's in } A^TA\cdot\underline{x}=0} \Rightarrow rk\,A = rk\,A^TA.$$

3.14.

$$\underline{w} = \begin{pmatrix} -4 \\ 3 \\ h \end{pmatrix} \qquad \underline{v_1} = \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix} \quad \underline{v_2} = \begin{pmatrix} 5 \\ -4 \\ -7 \end{pmatrix} \quad \underline{v_3} = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$$

$$x_1 \cdot \underline{v_1} + x_2 \underline{v_2} + x_3 \underline{v_3} = \underline{w}$$

$$\left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ -1 & -4 & 1 & 3 \\ -2 & -7 & 0 & h \end{array}\right) \begin{matrix} 1 \\ +2 \end{matrix} \rightarrow \left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 3 & -6 & h-8 \end{array}\right) -3$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 5 & -3 & -4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & h-5 \end{array}\right)$$

$h=5$: inf. many sol. $\underline{w}$ is in span$(\underline{v_1}, \underline{v_2}, \underline{v_3})$

$h \neq 5$: no solutions $\underline{w}$ is not in the span

4.3b)

$$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 1 & 1 \\ 2 & 0 & -2 \end{pmatrix}$$

Eigenvalues!

$$\begin{vmatrix} 2-\lambda & 1 & -1 \\ 0 & 1-\lambda & 1 \\ 2 & 0 & -2-\lambda \end{vmatrix} = 0$$

$$(2-\lambda) \cdot [(1-\lambda)(-2-\lambda)] + \underbrace{2(1+1-\lambda)}_{2(2-\lambda)}$$

$$(2-\lambda)(\lambda^2+\lambda-2) + 2(2-\lambda) = 0$$

$$(2-\lambda) \cdot (\lambda^2+\lambda) = 0$$

$$(2-\lambda) \cdot \lambda \cdot (\lambda+1) = 0$$

$$\lambda = 2, \quad \lambda = 0, \quad \lambda = -1$$

4.5. $A = \begin{pmatrix} 1 & 18 & 30 \\ -2 & -11 & -10 \\ 2 & 6 & 5 \end{pmatrix}$ $\quad \underline{v}_1 = \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix}$ $\underline{v}_2 = \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix}$ $\underline{v}_3 = \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$

$\boxed{A \cdot \underline{v} = \lambda \underline{v}}$

$$A \cdot \underline{v}_1 = \begin{pmatrix} 15 \\ -5 \\ 0 \end{pmatrix} = \underset{\underset{-5}{\parallel}}{\lambda} \cdot \begin{pmatrix} -3 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \lambda_1 = \underline{-5}$$

$$A \cdot \underline{v}_2 = \begin{pmatrix} 25 \\ 0 \\ -5 \end{pmatrix} = \underset{\underset{-5}{\parallel}}{\lambda} \cdot \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} \Rightarrow \lambda_2 = \underline{-5}$$

$$\begin{vmatrix} -3 & -5 \\ \boxed{\begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix}} \end{vmatrix} \neq 0$$

$\Rightarrow \underline{v}_1, \underline{v}_2$ lin. independent

$$A \cdot \underline{v}_3 = \begin{pmatrix} 15 \\ -5 \\ 5 \end{pmatrix} = 5 \cdot \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix} \Rightarrow \lambda_3 = 5$$

$\Downarrow$ A diag.

$$D = \underline{\begin{pmatrix} -5 & 0 & 0 \\ 0 & -5 & 0 \\ 0 & 0 & 5 \end{pmatrix}} \qquad P = \begin{pmatrix} \underline{v}_1 \,|\, \underline{v}_2 \,|\, \underline{v}_3 \end{pmatrix} = \underline{\begin{pmatrix} -3 & -5 & 3 \\ 1 & 0 & -1 \\ 0 & 1 & 1 \end{pmatrix}}$$

4.6. $A = \begin{pmatrix} 2 & -7 \\ 3 & -8 \end{pmatrix}$

$\begin{vmatrix} 2-\lambda & -7 \\ 3 & -8-\lambda \end{vmatrix} = 0$

$\lambda^2 - tr(A)\lambda + det(A) = 0$

$\lambda^2 + 6\lambda + 5 = 0$

$\lambda = \dfrac{-6 \pm \sqrt{36 - 4\cdot 5}}{2} = \dfrac{-6 \pm 4}{2}$

$\lambda = -1$, $\lambda = -5$

$\begin{pmatrix} 3 & -7 \\ 3 & -7 \end{pmatrix}$ $\quad 3x - 7y = 0$, y free

$x = \frac{7}{3}y$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 7/3\, y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 7/3 \\ 1 \end{pmatrix}$

$\frac{y}{3}\begin{pmatrix} 7 \\ 3 \end{pmatrix}$

$$\boxed{\begin{array}{l} v_1 = \begin{pmatrix} 7 \\ 3 \end{pmatrix} \\ E_{-1} = \text{span}(v_1) \end{array}}$$

$\begin{pmatrix} 7 & -7 \\ 3 & -3 \end{pmatrix}$

$v_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$E_{-5} = \text{span}(v_2)$

$P = \begin{pmatrix} 7 & 1 \\ 3 & 1 \end{pmatrix}$ or $P = \begin{pmatrix} 7/3 & 1 \\ 1 & 1 \end{pmatrix}$

4.9. $A = \begin{pmatrix} 4 & 1 & 2 \\ 0 & 3 & 0 \\ 1 & 1 & 5 \end{pmatrix}$

Eigenvalues: $\begin{vmatrix} 4-\lambda & 1 & 2 \\ 0 & 3-\lambda & 0 \\ 1 & 1 & 5-\lambda \end{vmatrix} = 0$

$(3-\lambda)\cdot\left((4-\lambda)(5-\lambda)-2\right)=0$

$\lambda=3$ or $\lambda^2-9\lambda+18=0$

$\lambda=3,\ \lambda=6$

Eigenvectors:

E_3: $(\lambda=3)$ $\begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ y, z free

$x=-y-2z$, $y=y$, $z=z$ } free

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -y-2z \\ y \\ z \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

Base: $v_1 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ $v_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$

E_6: $(\lambda=6)$ $\begin{pmatrix} -2 & 1 & 2 \\ 0 & -3 & 0 \\ 1 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & -3 & 0 \\ 0 & 3 & 0 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ z free

$x=-y+z=z$

$-3y=0 \Rightarrow y=0$

$z=z$ (free)

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ z \end{pmatrix} = z\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$ Base: $v_3 = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

A is diagonalizable: $D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{pmatrix}$ $P = \begin{pmatrix} -1 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$: $\boxed{P^{-1}AP = D}$

enough eigenvalues — enough eigenvectors — (from theory)

$P^{-1}AP = D \Rightarrow A = PDP^{-1} \Rightarrow A^{17} = \underbrace{(PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})\ldots(\cancel{P}DP^{-1})}_{17 \text{ times}} = P\cdot D^{17}\cdot P^{-1}$

$$P^{-1} = \frac{1}{|P|}\cdot \operatorname{adj}(P) = \frac{1}{3}\cdot \begin{pmatrix} 0 & -1 & 1 \\ 3 & -1 & 1 \\ 0 & 1 & 2 \end{pmatrix}^T = \frac{1}{3}\begin{pmatrix} 0 & 3 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$A^{17} = P D^{17} P^{-1} = \begin{pmatrix} -1 & -2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} \cdot \begin{pmatrix} 3^{17} & 0 & 0 \\ 0 & 3^{17} & 0 \\ 0 & 0 & 6^{17} \end{pmatrix} \cdot \frac{1}{3}\begin{pmatrix} 0 & 3 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= \begin{pmatrix} -3^{17} & -2\cdot 3^{17} & 6^{17} \\ 3^{17} & 0 & 0 \\ 0 & 3^{17} & 6^{17} \end{pmatrix} \cdot \frac{1}{3}\begin{pmatrix} 0 & 3 & 0 \\ -1 & -1 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= \frac{1}{3}\cdot \begin{pmatrix} 2\cdot 3^{17} + 6^{17} & -3\cdot 3^{17} + 2\cdot 3^{17} + 6^{17} & -2\cdot 3^{17} + 2\cdot 6^{17} \\ 0 & 3\cdot 3^{17} & 0 \\ -3^{17} + 6^{17} & -3^{17} + 6^{17} & 3^{17} + 2\cdot 6^{17} \end{pmatrix}$$

$$= 3^{17}\cdot \frac{1}{3}\cdot \begin{pmatrix} 2 & -1 & -2 \\ 0 & 3 & 0 \\ -1 & -1 & 1 \end{pmatrix} + \frac{1}{3}\cdot 6^{17}\cdot \begin{pmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

$$= 3^{16}\cdot \begin{pmatrix} 2 & -1 & -2 \\ 0 & 3 & 0 \\ -1 & -1 & 1 \end{pmatrix} + 6^{16}\cdot \begin{pmatrix} 2 & 2 & 4 \\ 0 & 0 & 0 \\ 2 & 2 & 4 \end{pmatrix}$$

4.10. $A = \begin{pmatrix} 1 & 7 & -2 \\ 0 & s & 0 \\ 1 & 1 & 4 \end{pmatrix}$ $\quad \underline{v} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

a) $|A| = s \cdot (4+2) = \underline{\underline{6s}}$

rk A: $s \neq 0 \Rightarrow \text{rk} A = 3$ (since $|A| \neq 0$)

$s = 0$: $\text{rk} A = 2$

$A = \begin{pmatrix} 1 & 7 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & 4 \end{pmatrix}$ since $M_{13,12} = 1 - 7 = -6 \neq 0$

$$\text{rk} A = \begin{cases} 3, & s \neq 0 \\ 2, & s = 0 \end{cases}$$

b) $\begin{vmatrix} 1-\lambda & 7 & -2 \\ 0 & s-\lambda & 0 \\ 1 & 1 & 4-\lambda \end{vmatrix} = 0$

$(s-\lambda) \cdot [(1-\lambda)(4-\lambda) + 2] = 0$

$\underline{\lambda_1 = s}$ or $\lambda^2 - 5\lambda + 6 = 0$

$\underline{\lambda_2 = 2}$, $\underline{\lambda_3 = 3}$

$A\underline{u} = \lambda \underline{u}$: $\begin{pmatrix} 1 & 7 & -2 \\ 0 & s & 0 \\ 1 & 1 & 4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 6 \\ s \\ 6 \end{pmatrix} = \lambda \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$

$\lambda = 6$, $\underline{s = 6}$

$\underline{u}$ is an eigenvector if and only if $s = 6$ (with $\lambda = 6$)

$$A = \begin{pmatrix} 1 & 7 & -2 \\ 0 & s & 0 \\ 1 & 1 & 4 \end{pmatrix}$$

c) Is A diagonalizable?

$\lambda_1 = s, \quad \lambda_2 = 2, \quad \lambda_3 = 3 \quad \Rightarrow \quad D = \begin{pmatrix} s & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$s \neq 2,3$: three eigenvalues of mult. 1

$\Rightarrow$ A diagonalizable

$s=2$:

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} -1 & 7 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & 2 \end{pmatrix}$$

$\lambda = 2$ mult. 2

need two free in E_2:

$M_{13,12} = -8 \neq 0$

one free var.

A not diag. $s=2$

$s=3$:

$$D = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix} \qquad A - \lambda I = \begin{pmatrix} -2 & 7 & -2 \\ 0 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

$\lambda = 3$ mult. 2

$M_{13,12} = -9 \neq 0$

one free var.

A not diag for $s=3$

A diag. $\Leftrightarrow s \neq 2,3$

C Midterm exam 10/2017

Exam	Midterm exam in GRA 6035 Mathematics
Date	October 13th, 2017 at 1500 - 1600

This exam has 8 questions

QUESTION 1.

Consider a 4×6 linear system $A \cdot \mathbf{x} = \mathbf{b}$, where the coefficient matrix A has a pivot position in every row. **Which statement is true?**

(a) The linear system has a unique solution
(b) The linear system is inconsistent
(c) The linear system has one degree of freedom
(d) The linear system has two degrees of freedom
(e) I prefer not to answer.

QUESTION 2.

Consider the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, given by

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 1 \\ 3 \\ t \end{pmatrix}$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & t \end{vmatrix}$$

Which statement is true?

$= 1\cdot(2t-12) - 1\cdot(t-3)$
$+ 1\cdot 2$
$= t-7$
indep. $t \neq 7$

(a) The vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent for all t
(b) The vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly dependent exactly when $t = 7$
(c) The vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent exactly when $t = 7$
(d) The vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent exactly when $t = 9$
(e) I prefer not to answer.

$$\begin{pmatrix} \boxed{\begin{matrix} -2 & 1 \\ 1 & -2 \end{matrix}} & \begin{matrix} 1 \\ 1 \end{matrix} \\ \begin{matrix} 1 & 1 \end{matrix} & -2 \end{pmatrix}$$

QUESTION 3.

Consider the matrix

$$A = \begin{pmatrix} t & 1 & 1 \\ 1 & t & 1 \\ 1 & 1 & t \end{pmatrix}$$

$|A| = t\cdot(t^2-1)$
$-1\cdot(t-1) + 1\cdot(1-t)$
$= t(t^2-1) + (t-1)\cdot(-2)$
$= t(t+1)(t-1) - 2(t-1)$
$= (t-1)[t(t+1)-2]$
$= (t-1)(t^2+t-2) = 0$
$t=1, \quad t=1, t=-2$
$t=1: rk=1$
$t=-2 \; rk=2$

Which statement is true?

(a) There is one value of t such that $\text{rk}(A) = 2$
(b) There are two values of t such that $\text{rk}(A) = 2$
(c) There are no values of t such that $\text{rk}(A) = 2$
(d) There are three values of t such that $\text{rk}(A) = 2$
(e) I prefer not to answer.

Question 4.

$$\begin{vmatrix} 3-\lambda & 5 & 2 \\ 0 & 2-\lambda & 0 \\ 1 & 3 & 4-\lambda \end{vmatrix} = 0$$

Consider the matrix

$$A = \begin{pmatrix} 3 & 5 & 2 \\ 0 & 2 & 0 \\ 1 & 3 & 4 \end{pmatrix}$$

$(2-\lambda)\cdot\left((3-\lambda)(4-\lambda)-2\right) = 0$

$\lambda = 2$ or $\lambda^2 - 7\lambda + 10 = 0$

$\lambda = 2, \lambda = 5$

Which statement is true?

(a) A has three distinct eigenvalues
(b) A has an eigenvalue of multiplicity two, and another eigenvalue of multiplicity one
(c) A has an eigenvalue of multiplicity three
(d) A has one eigenvalues of multiplicity one, and no other eigenvalues
(e) I prefer not to answer.

Question 5.

Consider the matrix A given by

$$A = \begin{pmatrix} 1 & s & 1 \\ 0 & 1 & s \\ 0 & 0 & -1 \end{pmatrix}$$

$\underbrace{\lambda_1 = \lambda_2 = 1}_{\text{mult 2}}, \underbrace{\lambda_3 = -1}_{\text{mult.1}}$

$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$

Which statement is true?

$\underline{E_1}$: $\begin{pmatrix} 0 & s & 1 \\ 0 & 0 & s \\ 0 & 0 & -2 \end{pmatrix}$

(a) A is diagonalizable for all s
(b) A is diagonalizable exactly when $s = 1$
(c) A is diagonalizable exactly when $s = 0$
(d) A is not diagonalizable for any s
(e) I prefer not to answer.

$\underline{s=0}$: $\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{pmatrix}$ 2 free

$\underline{s \neq 0}$: 1 free

A diag $\Leftrightarrow$ $\underline{s=0}$

Question 6.

Consider the quadratic form

$$f(x, y, z) = x^2 + 4xy + 2xz + 4y^2 + 4yz$$

Which statement is true?

(a) f is positive semidefinite but not positive definite
(b) f is positive definite
(c) f is negative definite
(d) f is indefinite
(e) I prefer not to answer.

Question 7.

Consider the function $f(x, y, z) = 1 - x^4 - 2x^2 + 4xz - y^2 - z^4 - 2z^2$. **Which statement is true?**

(a) f has a saddle point at $(x, y, z) = (0, 0, 0)$.
(b) f has a global maximum point at $(x, y, z) = (0, 0, 0)$.
(c) f has a global minimum point at $(x, y, z) = (0, 0, 0)$.
(d) f has a local maximum point at $(x, y, z) = (0, 0, 0)$, but no global maximum.
(e) I prefer not to answer.