Selected problems:

Key problems: 5.1cd, 5.3e, 6.1d, 6.2, 6.4de, 7.2

Workbook: 5.8, 5.9, [illegible], 5.15, 6.1c, 6.2a, 6.22

Midterm Exam: 10/2017 Q. 6-8
10/2015 Q. 1-8
01/2017 Q. 8
05/2017 Q. 5, 8

no time for these problems today, ask me in office hours if it is unclear how to do them

Key problems:

5.1 c) $A = \begin{pmatrix} 0.75 & 0.02 & 0.10 \\ 0.20 & 0.90 & 0.20 \\ 0.05 & 0.08 & 0.70 \end{pmatrix} \xrightarrow{\lambda=1} \begin{pmatrix} -0.25 & 0.02 & 0.10 \\ 0.20 & -0.10 & 0.20 \\ 0.05 & 0.08 & -0.30 \end{pmatrix} \begin{matrix} \cdot 100 \\ \cdot 100 \\ \cdot 100 \end{matrix}$

regular

$\rightarrow \begin{pmatrix} 5 & 8 & -30 \\ 20 & -10 & 20 \\ -25 & 2 & 10 \end{pmatrix} \rightarrow \begin{pmatrix} 5 & 8 & -30 \\ 0 & -42 & 140 \\ \cancel{0} & \cancel{42} & \cancel{-140} \end{pmatrix}$

z free

$-42y + 140z = 0 \iff y = \frac{140}{42}z = \frac{10}{3}z$

$5x + 8y - 30z = 0$
$5x = 30z - 8 \cdot \frac{10}{3}z$
$5x = \frac{90z - 80z}{3} = \frac{10}{3}z$
$x = \frac{2}{3}z$

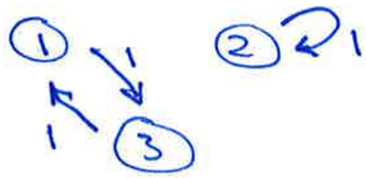
$E_1: \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 2/3\, z \\ 10/3\, z \\ z \end{pmatrix}$

$\frac{2}{3}z + \frac{10}{3}z + z = 1$
$5z = 1$
$z = 1/5$

$\boxed{\begin{aligned} x &= 2/15 \\ y &= 10/15 \\ z &= 3/15 \end{aligned}}$

equilibrium state = unique vector in E_1 that is a state vector

d) $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$



not regular

$\underline{x}_0 = \begin{pmatrix} a \\ b \\ c \end{pmatrix} \rightsquigarrow A^n \cdot \underline{x}_0$

$= \begin{cases} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} c \\ b \\ a \end{pmatrix}, & n \text{ odd} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, & n \text{ even} \end{cases}$

<u>Compute A^n:</u>

$A^2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \underline{I}$

$A^n = \begin{cases} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, & n \text{ odd} \\ \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, & n \text{ even} \end{cases}$

$a \neq c$: no equilibrium

$a = c$: $\underline{v} = \underline{\underline{\begin{pmatrix} a \\ b \\ a \end{pmatrix}}}$ eq. state

B.3 e) $f = xw - yz$

$\underline{A = \begin{pmatrix} 0 & 0 & 0 & 1/2 \\ 0 & 0 & -1/2 & 0 \\ 0 & -1/2 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{pmatrix}}$

$D_1 = 0$
$D_2 = 0$

$\Delta_1 = 0, 0, 0, 0$
$\Delta_2 = 0, 0, -1/4, \ldots$

f is <u>indefinite</u>

6.1 d) $f = x^2 + y^2 + z^2 + w^2 + xy + yz + zw$

$$A = \begin{pmatrix} 1 & 1/2 & 0 & 0 \\ 1/2 & 1 & 1/2 & 0 \\ 0 & 1/2 & 1 & 1/2 \\ 0 & 0 & 1/2 & 1 \end{pmatrix}$$

$D_1 = 1 > 0$

$D_2 = 3/4 > 0$

$D_3 = 1 \cdot 3/4 - 1/2 \cdot 1/2 = 2/4 = 1/2 > 0$

$D_4 = 1 \cdot 1/2 - 1/2 \cdot \begin{vmatrix} 1 & 1/2 & 0 \\ 1/2 & 1 & 0 \\ 0 & 1/2 & 1/2 \end{vmatrix}$

$= \frac{1}{2} - \frac{1}{2}\left(\frac{1}{2} \cdot \frac{3}{4} - \cancel{\frac{1}{2} \cdot 0}\right)$

$= \frac{1}{2} - \frac{3}{16} = \frac{5}{16} > 0$

f is positive defn.

6.2

$$A = \begin{pmatrix} a & 0 & 0 & -1 \\ 0 & a & -1 & 0 \\ 0 & -1 & a & 0 \\ -1 & 0 & 0 & a \end{pmatrix}$$

$\Delta_1 = a, a, a, a \leq 0$

$\Delta_2 = a^2, a^2, a^2-1,$
$a^2-1, a^2, a^2 \geq 0$

$\Delta_3 = a \cdot (a^2-1), a(a^2-1),$
$a(a^2-1), a(a^2-1) \leq 0$

$\Delta_4 = a \cdot a(a^2-1) + 1 \cdot (-1)(a^2-1)$
$= a^2 \cdot (a^2-1) - 1 \cdot (a^2-1)$
$= (a^2-1) \cdot (a^2-1) = (a^2-1)^2 \geq 0$

A neg. semidefn.

$\Updownarrow$

all $\Delta_1 \leq 0$

" $\Delta_2 \geq 0$

" $\Delta_3 \leq 0$

" $\Delta_4 \geq 0$

Conditions:

$\left. \begin{array}{l} a \leq 0 \\ a^2 - 1 \geq 0 \end{array} \right\}$ $a \leq -1$

6.4 d) $f = 16 - x^4 - 2x^2 - 3y^2 + 6xz - 6z^2$

$f'_x = -4x^3 - 4x + 6z$

$f'_y = -6y$

$f'_z = 6x - 12z$

$$H(f) = \begin{pmatrix} -12x^2-4 & 0 & 6 \\ 0 & -6 & 0 \\ 6 & 0 & -12 \end{pmatrix}$$

f concave
$\Updownarrow$
$H(f)$ neg. semidefn. for all (x,y,z)

$D_1 = -12x^2 - 4 < 0$

$D_2 = -6(-12x^2-4) > 0$

$D_3 = -6 \cdot (-12(-12x^2-4) - 36)$

$= -6(144x^2 + 48 - 36)$

$= -6(144x^2 + 12) < 0$

for all x,y,z

f is concave

e) $f = \dfrac{xy + xz + yz}{xyz}$, $x,y,z > 0$

$= \dfrac{1}{z} + \dfrac{1}{y} + \dfrac{1}{x}$

$f'_x = -\dfrac{1}{x^2}$

$f'_y = -\dfrac{1}{y^2}$

$f'_z = -\dfrac{1}{z^2}$

$$H(f) = \begin{pmatrix} \frac{2}{x^3} & 0 & 0 \\ 0 & 2/y^3 & 0 \\ 0 & 0 & 2/z^3 \end{pmatrix}$$

$\lambda_1 = 2/x^3 > 0$

$\lambda_2 = 2/y^3 > 0$

$\lambda_3 = 2/z^3 > 0$

for all $x,y,z > 0$

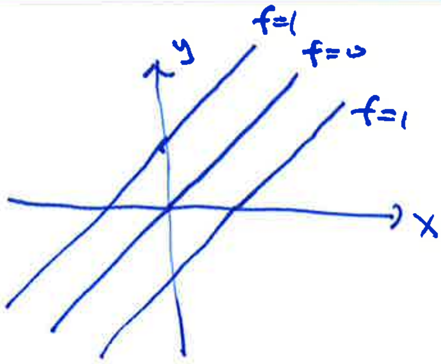
f is convex

$D_1 = 2/x^3 > 0$

$D_2 = 4/x^3y^3 > 0$

$D_3 = 8/x^3y^3z^3 > 0$

7.2. $f(x,y) = |x-y|$



convex set

graph of f

f <u>convex function</u>

$$g(x,y,z) = 1 - e^{x-y+z} = 1 - e^u, \quad u = x-y+z$$

$$g'_x = -e^u \cdot u'_x = -e^u \cdot 1 = \underline{-e^u}$$

$$g'_y = -e^u \cdot u'_y = \underline{e^u}$$

$$g'_z = -e^u \cdot u'_z = -e^u$$

$$H(g) = \begin{pmatrix} -e^u & +e^u & -e^u \\ +e^u & -e^u & +e^u \\ -e^u & +e^u & -e^u \end{pmatrix} = \begin{pmatrix} -e^u & e^u & -e^u \\ e^u & -e^u & e^u \\ -e^u & e^u & -e^u \end{pmatrix}$$

rank 1

$$D_1 = -e^u < 0$$

$$D_2 = (e^u)^2 - (e^u)^2 = 0$$

<u>RRC:</u> $D_1 < 0$

$\Rightarrow$ $H(g)$ <u>neg. semidefn.</u> for all x,y,z

g <u>concave</u>

Midterm 10/2017:

6.)

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 2 & 4 & 2 \\ 1 & 2 & 0 \end{pmatrix}$$

$D_1 = 1$

$D_2 = 0$

$D_3 = 1 \cdot 0 - 2 \cdot 0 = 0$

A indefinite (D)

$\Delta_1 = 1, 4, 0$

$\Delta_2 = 0, \underline{-4}$

$$H(f) = \begin{pmatrix} 2 & 4 & 2 \\ 4 & 8 & 4 \\ 2 & 4 & 0 \end{pmatrix}$$

$D_1 = 2$

$\Delta_2 = -16$

7) $f = 1 - x^4 - 2x^2 + 4xz - y^2 - z^4 - 2z^2$

$f'_x = -4x^3 - 4x + 4z = 0$

$f'_y = -2y = 0$

$f'_z = 4x - 4z^3 - 4z = 0$

(0,0,0) is a stationary pt.

$$H(f) = \begin{pmatrix} -12x^2-4 & 0 & 4 \\ 0 & -2 & 0 \\ 4 & 0 & -12z^2-4 \end{pmatrix}$$

(B)

(0,0,0) global max

$\Uparrow$

f concave

$\Uparrow$

$D_1 = -12x^2 - 4 < 0$ for all x,y,z

$D_2 = -2(-12x^2-4) > 0$ — " —

$D_3 = -2 \cdot ((-12x^2-4)(-12z^2-4) - 16)$

$= -2(144x^2z^2 + 48x^2 + 48z^2) \leq 0$ — " —

neg. semidefin. for all x,y,z

$D_1 < 0$, $D_2 > 0$, $D_3 < 0$ or $D_1 < 0$, $D_2 > 0$, $D_3 = 0$

RRL

8.) $A = \begin{pmatrix} 0.80 & 0.20 \\ 0.20 & 0.80 \end{pmatrix}$

regular

$\underline{E_1}$: $\lambda = 1$

$$\begin{pmatrix} -0.20 & 0.20 \\ \cancel{0.20} & \cancel{-0.20} \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$-0.2x + 0.2y = 0$

y free

$\underline{x = y}$

$\underline{E_1}$: $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$y + y = 1$

$2y = 1$

$y = 1/2$

$\boxed{\begin{matrix} x = 1/2 \\ y = 1/2 \end{matrix}}$

C

Midterm 10/2015.

5.) $A = \begin{pmatrix} 1 & 5 & 5 \\ 0 & 2 & 5 \\ 0 & 0 & 3 \end{pmatrix}$

$\lambda_1 = 1$

$\lambda_2 = 2$

$\lambda_3 = 3$

diagonalizable for all s since all eigenval. have mult. I

A

6.) $A = \begin{pmatrix} 0.74 & 0.13 \\ 0.26 & 0.87 \end{pmatrix}$

regular

$\underline{E_1}$: $\begin{pmatrix} -0.26 & 0.13 \\ \cancel{0.26} & \cancel{-0.13} \end{pmatrix}$

$-0.26x + 0.13y = 0$

y free

$$\frac{\cancel{-0.26}x}{\cancel{-0.26}} = \frac{-0.13y}{-0.26}$$

$x = \frac{1}{2}y$

$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y/2 \\ y \end{pmatrix}$

$y/2 + y = 1$

$\frac{3}{2}y = 1$

$\underline{y = 2/3}$

$\boxed{\begin{matrix} x = 1/3 \\ y = 2/3 \end{matrix}}$

eq. state

D

7

$$A = \begin{pmatrix} 2 & 3 & 0 & 0 \\ 3 & 5 & -1 & 0 \\ 0 & -1 & 3 & 1 \\ 0 & 0 & 1 & 4 \end{pmatrix}$$

$D_1 = 2$

$D_2 = 1$

$D_3 = 3 \cdot 1 + 1 \cdot (-2) = 1$

$D_4 = 4 \cdot 1 - 1 \cdot \begin{vmatrix} 2 & 3 & 0 \\ 3 & 5 & 0 \\ 0 & -1 & 1 \end{vmatrix}$

$= 4 - 1 \cdot (10 - 9) = 3$

f pos. defn.

B

8. $f = x^a \cdot \sqrt{y} = x^a y^{1/2}$, $x, y > 0$

$a > 0$

$f'_x = a x^{a-1} \cdot \sqrt{y} = a x^{a-1} y^{1/2}$

$f'_y = x^a \cdot \frac{1}{2} y^{-1/2} = \frac{1}{2} x^a y^{-1/2}$

$f''_{xx} = a(a-1) x^{a-2} \cdot y^{1/2}$

$f''_{xy} = \frac{1}{2} a \cdot x^{a-1} y^{-1/2}$

$f''_{yy} = -\frac{1}{4} x^a y^{-3/2}$

$H(f) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{xy} & f''_{yy} \end{pmatrix}$

$D_1 = a(a-1) x^{a-2} y^{1/2}$

↑ same sign as $a-1$

$0 < a < 1 : D_1 < 0$

$a > 1 : D_1 > 0$

$D_2 = -\frac{1}{4} a(a-1) \cdot x^{2a-2} y^{-1} - \left(\frac{1}{2} a x^{a-1} y^{-1/2}\right)^2$

$= -\frac{1}{4} a(a-1) x^{2a-2} y^{-1} - \frac{1}{4} a^2 x^{2a-2} y^{-1}$

$= -\frac{1}{4} a x^{2a-2} y^{-1} (a - \cancel{1} + \cancel{a}) = -\frac{1}{4} a^2 x^{2a-2} y^{-1} (2a-1)$

$= \frac{1}{4} a^2 x^{2a-2} y^{-1} (1-2a) \geq 0$

$\Updownarrow$

$a \leq 1/2$

Cond:

$0 < a \leq 1/2$: neg. semi-defn.

C

f concave

Workbook:

6.1 d) $f = xy^2 + x^3y - xy$

$f'_x = y^2 + 3x^2y - y = 0 \qquad y(y+3x^2-1)=0$

$f'_y = 2xy + x^3 - x = 0 \qquad x(2y+x^2-1)=0$

$y=0$ or $y+3x^2-1=0$

and

$x=0$ or $2y+x^2-1=0$

a) $(0,0)$

b) $y=0,\ 2y+x^2-1=0$
$x=\pm 1 \Rightarrow (\pm 1, 0)$

c) $y+3x^2-1=0,\ x=0$
$y=1 \Rightarrow (0,1)$

d) $y+3x^2-1=0,\ 2y+x^2-1=0$

$y = 1-3x^2 \rightarrow 2(1-3x^2)+x^2-1=0$

$2-6x^2+x^2-1=0$

$-5x^2+1=0$

$x^2 = 1/5$

$x = \pm\sqrt{1/5} \Rightarrow (\pm\sqrt{1/5}, 2/5)$

$y = 1-3\cdot 1/5 = 2/5$

$$H(f) = \begin{pmatrix} 6xy & 2y+3x^2-1 \\ 2y+3x^2-1 & 2x \end{pmatrix}$$

$D_1 = 6xy$

$D_2 = 12x^2y - (2y+3x^2-1)^2$

$(0,0)$: $D_1=0, D_2=-1$ saddle pt.

$(\pm1,0)$: $D_1=0, D_2=-4$ — " —

$(0,1)$: $D_1=0, D_2=-1$ — " —

$(\pm\sqrt{1/5}, 2/5)$: $D_1 = \pm \frac{12}{5}\cdot\sqrt{1/5}$

$D_2 = 12\,\frac{1}{5}\cdot\frac{2}{5} - (\frac{4}{5}+\frac{3}{5}-1)^2 = 24/25 - 4/25 = 20/25 > 0$

$(+\sqrt{1/5}, 2/5)$: local min

$(-\sqrt{1/5}, 2/5)$: local max

6.2 a) $f = x^2 + 6xy + y^2 - 3yz + 4z^2 - 10x - 5y - 21z$

$f'_x = 2x + 6y - 10 = 0$

$f'_y = 6x + 2y - 3z - 5 = 0$

$f'_z = -3y + 8z - 21 = 0$

$$\boxed{\begin{aligned} 2x + 6y &= 10 \\ 6x + 2y - 3z &= 5 \\ -3y + 8z &= 21 \end{aligned}}$$

Gauss: $(x,y,z) = (3,1,2)$

$H(f) = \begin{pmatrix} 2 & 6 & 0 \\ 6 & 2 & -3 \\ 0 & -3 & 8 \end{pmatrix}$

$D_1 = 2$

$D_2 = 4 - 36 = -32 < 0$

$D_3 = 8(-32) + 3(-6) \neq 0$

indefinite

saddle pt.

6.22.

$f = 3 - a \cdot Q(x_1, x_2)$

Q pos. defn.

$H(f) = -a \cdot H(Q)$

$\uparrow$

pos. defn.

$a > 0$: $H(f)$ neg. defn.

$a < 0$: $H(f)$ pos. defn.

D