GRA 6035 Mathematics

Problems for Lecture 1

There are **exercise sessions** on Mon Aug 27 at 17-19 and Thu Aug 30 at 18-20 in C2-020, see the plan for exercise sessions for details. The **key problems** in this problem sheet are intended for the exercise sessions (answers are given below, and you will get help in the exercise sessions). After doing the key problems, you can start on the problems in the Workbook (you will find full solutions there).

Key problems

Problem 1.
Use Gaussian elimation to solve the linear systems

$$a)\quad \begin{aligned} x + y + z &= 11 \\ x - y + z &= 9 \\ x + 2y + 4z &= 33 \end{aligned} \qquad b)\quad \begin{aligned} x + y + z &= 11 \\ x - y + z &= 9 \\ x + 2y + 4z &= 33 \\ 3x - y + 2z &= 45 \end{aligned} \qquad c)\quad \begin{aligned} x + y + z + w &= 10 \\ x - y + z + 11w &= 16 \\ x + 2y + 4z - w &= 7 \end{aligned}$$

Problem 2.
Determine how many solutions the linear system has:

$$\begin{aligned} x + y + z &= 6 \\ x + 2y + 4z &= 13 \\ x + 3y + 9z &= 24 \end{aligned}$$

Does the number of solutions change if we change the blue coefficient in the second equation? In that case, determine how the number of solutions changes with the blue coefficient.

Problem 3.
Find the rank of the matrix A, given by

$$A = \begin{pmatrix} 1 & 7 & -3 & 5 & 10 \\ 2 & -3 & 1 & 4 & 18 \\ 1 & 24 & -10 & 11 & 12 \end{pmatrix}$$

Does the rank change if we change the red coefficient in the first row? In that case, determine how the rank of A changes with the red coefficient.

Problems from the Digital Workbook

Exercise problems	1.1 - 1.15 (full solutions in the workbook)
Exam problems	1.16 - 1.18 (full solutions in the workbook)

Solutions to key problems

Problem 1.
a) $(x, y, z) = (3, 1, 7)$ *b)* No solutions *c)* $(x, y, z, w) = (13 - 5w, 5w - 3, -w, w)$ with w free

Problem 2.
There is one unique solution. The number of solutions only changes if the blue coefficient is 5, in which case there are no solutions.

Problem 3.
The rank is 2. The rank is 3 for all values of the red coefficient except 1.

Solutions: Key problems Lecture 1

1. a)

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 11 \\ 1 & -1 & 1 & 9 \\ 1 & 2 & 4 & 33 \end{array}\right) \cdot(-1),\ \cdot(-1) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 11 \\ 0 & -2 & 0 & -2 \\ 0 & 1 & 3 & 22 \end{array}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 11 \\ 0 & 1 & 3 & 22 \\ 0 & -2 & 0 & -2 \end{array}\right) \cdot 2$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 11 \\ 0 & 1 & 3 & 22 \\ 0 & 0 & 6 & 42 \end{array}\right)$$

$x+1+7=11 \Rightarrow x=\underline{3}$

$y+3\cdot 7=22 \Rightarrow y=\underline{1}$

$6z=42 \Rightarrow z=\underline{7}$

one solution:

$(x,y,z)=\underline{\underline{(3,1,7)}}$

b)

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 11 \\ 1 & -1 & 1 & 9 \\ 1 & 2 & 4 & 33 \\ 3 & -1 & 2 & 45 \end{array}\right) \xrightarrow{\text{see above}} \left(\begin{array}{ccc|c} 1 & 1 & 1 & 11 \\ 0 & 1 & 3 & 22 \\ 0 & 0 & 6 & 42 \\ 3 & -1 & 2 & 45 \end{array}\right) \cdot(-3) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 11 \\ 0 & 1 & 3 & 22 \\ 0 & 0 & 6 & 42 \\ 0 & -4 & -1 & 12 \end{array}\right) \cdot 4$$

$$\rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 11 \\ 0 & 1 & 3 & 22 \\ 0 & 0 & 6 & 42 \\ 0 & 0 & 11 & 100 \end{array}\right) \cdot\left(-\tfrac{11}{6}\right) \rightarrow \left(\begin{array}{ccc|c} 1 & 1 & 1 & 11 \\ 0 & 1 & 3 & 22 \\ 0 & 0 & 6 & 42 \\ 0 & 0 & 0 & 23 \end{array}\right)$$

no solutions

c)

$$\left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10 \\ 1 & -1 & 1 & 11 & 16 \\ 1 & 2 & 4 & -1 & 7 \end{array}\right) \cdot(-1),\ \cdot(-1) \rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10 \\ 0 & -2 & 0 & 10 & 6 \\ 0 & 1 & 3 & -2 & -3 \end{array}\right) \cdot\tfrac{1}{2}$$

$$\rightarrow \left(\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 10 \\ 0 & -2 & 0 & 10 & 6 \\ 0 & 0 & 3 & 3 & 0 \end{array}\right)$$

w free, infinitely many solutions

$x+(5w-3)+(-w)+w=10 \Rightarrow x=\underline{-5w+13}$

$-2y+10w=6 \Rightarrow y=\underline{5w-3}$

$3z+3w=0 \Rightarrow z=\underline{-w}$

$(x,y,z,w)=\underline{\underline{(-5w+13,\ 5w-3,\ -w,\ w)}}$

with w free

2.

$$\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & a & 13 \\ 1 & 3 & 9 & 24 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & a-1 & 7 \\ 0 & 2 & 8 & 18 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & a-1 & 7 \\ 0 & 0 & 10-2a & 4 \end{array}\right)$$

$a=4$: $10-2a=2\neq 0 \Rightarrow$ one solution

$a=5$: $10-2a=0$, pivot in last col. (4) $\Rightarrow$ no solutions

$a\neq 5$: $10-2a\neq 0 \Rightarrow$ one solution.

Concl: one solution (for $a=4$), changes to no solutions if $a=5$, otherwise there is one solution

3.

$$\begin{pmatrix} a & 7 & -3 & 5 & 10 \\ 2 & -3 & 1 & 4 & 18 \\ 1 & 24 & -10 & 11 & 12 \end{pmatrix} \to \begin{pmatrix} 1 & 24 & -10 & 11 & 12 \\ 2 & -3 & 1 & 4 & 18 \\ a & 7 & -3 & 5 & 10 \end{pmatrix}$$

$$\to \begin{pmatrix} 1 & 24 & -10 & 11 & 12 \\ 0 & -51 & 21 & -18 & -6 \\ 0 & 7-24a & 10a-3 & 5-11a & 10-12a \end{pmatrix}$$

rk $A \geq 2$ and

rk $A = 2$ if and only if (row 3) $= c\cdot$(row 2) for some constant c

$$\Updownarrow$$

$$\frac{7-24a}{-51} = \frac{10a-3}{21} = c$$

$$21\cdot(7-24a) = -51(10a-3)$$
$$147-504a = -510a+153$$
$$6a = 6$$
$$\underline{a=1}$$

Concl:

If $a\neq 1$, then (row 3) $\neq c$(row 2), hence rk $A = 3$

If $a=1$, we get rk $A=2$ since:

$$\begin{pmatrix} 1 & 24 & -10 & 11 & 12 \\ 0 & -51 & 21 & -18 & -6 \\ 0 & -17 & 7 & -6 & -2 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 24 & -10 & 11 & 12 \\ 0 & -51 & 21 & -18 & -6 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$