GRA 6035 MATHEMATICS

Problems for Lecture 10

Key problems

Problem 1.
Solve the differential equations:
a) $y' = 3t^2 + 2$ *b)* $ty' = 1$ $y' = t\sqrt{t^2+1}$

Problem 2.
Solve the differential equations:
a) $y' = 5y$ *b)* $y' = y^2 t$ *c)* $y' = 5y(1 - y/10)$

Problem 3.
Solve the differential equations:
a) $y' + 3y = 6$ *b)* $y' - 2ty = 4t$ *c)* $y' + 2y = e^t$

Problems from Differential Equations

Exercise problems	1.1 - 1.16 (full solutions on the web page)

Problems from the Digital Workbook

Exercise problems	10.1 - 10.12 (full solutions in the workbook)
Excel problems	10.17 - 10.18 (full solutions in the workbook)
	As a minimum, you should understand what happens when you change the parameters in the Excel models that are available in the workbook.

Answers to key problems

Problem 1.
a) $y = t^3 + 2t + C$ *b)* $y = \ln|t| + C$ *c)* $y = \frac{1}{3}(t^2+1)\sqrt{t^2+1} + C$

Problem 2.
a) $y = Ke^{5t}$ *b)* $y = -2/(t^2 + 2C)$ *c)* $y = 10 \cdot Ke^{5t}/(1 + Ke^{5t})$

Problem 3.
a) $y = 2 + Ce^{-3t}$ *b)* $y = -2 + Ce^{t^2}$ *c)* $y = \frac{1}{3}e^t + Ce^{-2t}$

Solutions: Key problems Lecture 10

1. a) $y' = 3t^2 + 2$

$y = \int 3t^2 + 2\,dt = \underline{t^3 + 2t + C}$

b) $ty' = 1$

$y' = 1/t$

$y = \int 1/t\,dt = \underline{\ln|t| + C}$

c) $y' = t\sqrt{t^2+1}$

$y = \int t\sqrt{t^2+1}\,dt = \int \cancel{t}\sqrt{u}\cdot\frac{du}{2\cancel{t}} = \int \frac{1}{2}u^{1/2}\,du$

$u = t^2+1$
$du = 2t\,dt$

$u \cdot \sqrt{u}$

$= \frac{1}{\cancel{2}}\left(\frac{\cancel{2}}{3}u^{3/2}\right) + C = \frac{1}{3}u^{3/2} + C = \underline{\underline{\frac{1}{3}(t^2+1)\sqrt{t^2+1} + C}}$

2. a) $y' = 5y$

$\frac{1}{y}y' = 5$

$\int \frac{1}{y}\underbrace{y'\,dt}_{dy} = \int 5\,dt$

$\int \frac{1}{y}\,dy = \int 5\,dt$

$\ln|y| = 5t + C$

have combined C_1, C_2 in the two integrals

$|y| = e^{5t+C} = e^C \cdot e^{5t}$

$y = \pm e^C e^{5t} = Ke^{5t}$

$y = \underline{\underline{Ke^{5t}}}$

b) $y' = y^2 t$

$\frac{1}{y^2}y' = t$

$\int \frac{1}{y^2}y'\,dt = \int t\,dt$

$\int y^{-2}\,dy = \int t\,dt$

$-y^{-1} = \frac{1}{2}t^2 + C$

$-\frac{1}{y} = \frac{1}{2}t^2 + C$

$\frac{1}{y} = -\frac{1}{2}t^2 - C$

$y = \frac{1}{-\frac{1}{2}t^2 - C}\begin{matrix}(-2)\\(-2)\end{matrix}$

$= \underline{\frac{-2}{t^2 + 2C}}$

c) $y' = 5y(1 - y/10) = 5y \cdot \frac{1}{10} \cdot (10-y)$

$$\frac{1}{y(10-y)} y' = 5 \cdot \frac{1}{10} = \frac{1}{2}$$

$$\int \frac{1}{y(10-y)} y' dt = \int \frac{1}{2} dt$$

$$\int \frac{1}{y(10-y)} dy = \frac{1}{2} t + C$$

$$\int \frac{1/10}{y} + \frac{1/10}{10-y} dy = \frac{1}{2} t + C$$

$(10-y)' = -1$

$$\frac{1}{10} \ln|y| - \frac{1}{10} \ln|10-y| = \frac{1}{2} t + C$$

$$\ln|y| - \ln|10-y| = 10\left(\frac{1}{2}t + C\right)$$

$$\ln\left|\frac{y}{10-y}\right| = 5t + 10C$$

$$\left|\frac{y}{10-y}\right| = e^{5t+10C}$$

$$\frac{y}{10-y} = \pm e^{10C} e^{5t} = Ke^{5t}$$

$$y = Ke^{5t} \cdot (10-y)$$

$$y = 10Ke^{5t} - Ke^{5t} \cdot y$$

$$y + Ke^{5t} y = 10Ke^{5t}$$

$$\underline{\underline{y = \frac{10Ke^{5t}}{1 + Ke^{5t}}}}$$

$$\frac{1}{y(10-y)} = \frac{A}{y} + \frac{B}{10-y} \quad \Big| \cdot y(10-y)$$

$$1 = A \cdot (10-y) + B \cdot y$$

$$= 10A - Ay + By$$

$$1 = 10A + (B-A)y$$

$\Downarrow$ compare coeff.

$B - A = 0$ (no y-term on LHS)

$10A = 1$ (const. term)

$\Downarrow$

$\underline{A = \frac{1}{10}}$ $\quad B = A = \underline{\frac{1}{10}}$

$$\frac{1}{y(10-y)} = \frac{1/10}{y} + \frac{1/10}{10-y}$$

3\. a) $y' + 3y = 6$

Alt A: integrating factor $\quad y' + a(t)y = b(t)$

$a(t) = 3 \Rightarrow \int a(t)dt = 3t + C \quad \Rightarrow u = \underline{\underline{e^{3t}}}$

$(y' + 3y)e^{3t} = 6e^{3t}$

$(y \cdot e^{3t})' = 6e^{3t}$ follows from choice of u (theory)

$$y e^{3t} = \int 6e^{3t} dt = 6 \cdot \frac{1}{3}e^{3t} + C = 2e^{3t} + C$$

$$y = \frac{2e^{3t} + C}{e^{3t}} = 2 + \frac{C}{e^{3t}} = \underline{\underline{2 + C \cdot e^{-3t}}}$$

Alt B: superposition principle

$y = y_h + y_p = \underline{\underline{C \cdot e^{-3t} + 2}}$

y_h: $y' + 3y = 0$

$r + 3 = 0$

$r = -3 \Longrightarrow y_h = C \cdot e^{-3t}$

y_p: $y' + 3y = 6 \quad$ Try: $y = A$ (const.)

$y' = 0$

$\Downarrow$

$y' + 3y = 6$

$0 + 3A = 6$

$\underline{A = 2} \Rightarrow y_p = 2$

b) $y' - 2ty = 4t$ $\qquad u = e^{\int -2t\,dt} = e^{-t^2+C} = \underline{e^{-t^2}}$

$$(ye^{-t^2})' = 4te^{-t^2}$$

$$ye^{-t^2} = \int 4te^{-t^2}\,dt = \int 4\cancel{t}e^{u}\frac{du}{-2\cancel{t}} = -2\int e^{u}du$$

$$\boxed{\begin{array}{c} u = -t^2 \\ du = -2t\,dt \end{array}}$$

$$ye^{-t^2} = -2e^{u} + C = -2e^{-t^2} + C$$

$$y = \frac{-2e^{-t^2} + C}{e^{-t^2}} = -2 + \frac{C}{e^{-t^2}} = \underline{\underline{-2 + Ce^{t^2}}}$$

c) $y' + 2y = e^t$

<u>Alt A:</u> $u = e^{\int 2dt} = e^{2t}$

$$(ye^{2t})' = e^t \cdot e^{2t} = e^{3t}$$

$$ye^{2t} = \int e^{3t}dt = \frac{1}{3}e^{3t} + C$$

$$y = \frac{1}{3}\frac{e^{3t}}{e^{2t}} + \frac{C}{e^{2t}} = \underline{\underline{\frac{1}{3}e^{t} + Ce^{-2t}}}$$

<u>Alt B:</u>

$$y = y_h + y_p = \underline{\underline{Ce^{-2t} + \frac{1}{3}e^{t}}}$$

<u>y_h:</u> $y' + 2y = 0$

$r + 2 = 0$

$r = -2$ $\Rightarrow y_h = Ce^{-2t}$

<u>y_p:</u> $y' + 2y = e^t$

<u>Try:</u> $y = Ae^t$ (A const.)

$y' = Ae^t$

$y' + 2y = e^t \Rightarrow Ae^t + 2(Ae^t) = e^t$

$3Ae^t = e^t$

$3A = 1 \Rightarrow A = 1/3$

$A = 1/3$

$y_p = \frac{1}{3}e^t$