GRA 6035 Mathematics

Problems for Lecture 2

Key problems

Problem 1.
Compute the determinant of these matrices:

$$a)\quad A = \begin{pmatrix} 1 & 2 & 5 \\ 3 & 1 & 2 \\ 1 & 2 & 4 \end{pmatrix} \qquad b)\quad A = \begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix} \qquad c)\quad A = \begin{pmatrix} 1 & 0 & 0 & 3 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 0 & 0 & 1 \end{pmatrix}$$

Problem 2.
Use minors to determine the rank of these matrices. Can you find the pivot positions based on the minors?

$$a)\quad A = \begin{pmatrix} 4 & 1 & 1 & 3 & 7 \\ 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 1 & 0 \end{pmatrix} \qquad b)\quad A = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 2 & -1 & 7 & 3 \\ 4 & 5 & 11 & 10 \end{pmatrix} \qquad c)\quad A = \begin{pmatrix} 3 & 0 & 0 & 7 \\ 0 & 5 & 5 & 0 \\ 0 & 5 & 5 & 0 \end{pmatrix}$$

Problem 3.
Use minors to find the rank of these matrices:

$$a)\quad A = \begin{pmatrix} 1 & 3 & t \\ 2 & 5 & 7 \end{pmatrix} \qquad b)\quad A = \begin{pmatrix} a & 7 & -3 & 5 & 10 \\ 2 & -3 & 1 & 4 & 18 \\ 1 & 24 & -10 & 11 & 12 \end{pmatrix} \qquad c)\quad A = \begin{pmatrix} 1 & a & b \\ a & b & 1 \end{pmatrix}$$

Problem 4.
Use minors to determine the number of solutions of these linear systems. What are the possible choices of free variables, if any?

$$a)\quad \begin{aligned} x + y + z &= 6 \\ x + 2y + tz &= 13 \\ x + 3y + 9z &= 24 \end{aligned} \qquad b)\quad \begin{aligned} x + 4y + 5z - 3w &= 6 \\ 2x + 7y + z \phantom{{}- 3w} &= 4 \\ x + 5y + 4z - 8w &= 1 \end{aligned}$$

Problems from the Digital Workbook

Exercise problems	2.1 - 2.22 (full solutions in the workbook)
Exam problems	2.23 - 2.25 (full solutions in the workbook)

Answers to key problems

Problem 1.
a) $|A| = 5$ b) $|A| = 1 - a^2 - b^2 - c^2 + 2abc$ c) $|A| = -96$

Problem 2.

a) $\operatorname{rk} A = 3$ with $A = \begin{pmatrix} 4 & 1 & 1 & 3 & 7 \\ 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 1 & 0 \end{pmatrix}$ b) $\operatorname{rk} A = 3$ with $A = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 2 & -1 & 7 & 3 \\ 4 & 5 & 11 & 10 \end{pmatrix}$ c) $\operatorname{rk} A = 2$ with $A = \begin{pmatrix} 3 & 0 & 0 & 7 \\ 0 & 5 & 5 & 0 \\ 0 & 5 & 5 & 0 \end{pmatrix}$

Problem 3.
a) $\operatorname{rk} A = 2$ for all t b) $\operatorname{rk} A = 2$ if $a = 1$, and $\operatorname{rk} A = 3$ if $a \neq 1$ c) $\operatorname{rk} A = 1$ if $(a, b) = (1, 1)$ and $\operatorname{rk} A = 2$ if $(a, b) \neq (1, 1)$

Problem 4.
a) One solution if $t \neq 5$, and no solutions if $t = 5$ b) Infinitely many solutions (one degree of freedom).
In b) we may choose x, y, z or w as a free variable.

Solutions: Key problems Lecture 2

1. a) $\begin{vmatrix} 1 & 2 & 5 \\ 3 & 1 & 2 \\ 1 & 2 & 4 \end{vmatrix} = 1\cdot(4-4) - 2\cdot(12-2) + 5(6-1) = -20+25 = \underline{5}$

b) $\begin{vmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{vmatrix} = 1(1-c^2) - a(a-bc) + b(ac-b) = \underline{1 - a^2 - b^2 - c^2 - 2abc}$

c) $\begin{vmatrix} 1 & 0 & 0 & 3 \\ 0 & 4 & 2 & 0 \\ 0 & 2 & 4 & 0 \\ 3 & 0 & 0 & 1 \end{vmatrix} = 1\cdot\begin{vmatrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{vmatrix} - 3\begin{vmatrix} 0 & 4 & 2 \\ 0 & 2 & 4 \\ 3 & 0 & 0 \end{vmatrix} = 1\cdot 1\cdot(16-4) - 3\cdot 3\cdot(16-4)$

$= 1\cdot 12 - 9\cdot 12 = \underline{-96}$

2. a) $A = \begin{pmatrix} 4 & 1 & 1 & 3 & 7 \\ 2 & 1 & 0 & 1 & 0 \\ 1 & 0 & 3 & 1 & 0 \end{pmatrix}$

pivot pos.

$M_{123,123} = 1\cdot(-1) + 3\cdot 2 = 5 \neq 0 \Rightarrow \text{rk}\,A = \underline{3}$

b) $A = \begin{pmatrix} 1 & 3 & 2 & 4 \\ 2 & -1 & 7 & 3 \\ 4 & 5 & 11 & 10 \end{pmatrix}$

not pivot pos   pivot pos.

$M_{123,123} = 1\cdot(-11-35) - 3(22-28) + 2\cdot(10+4)$
$= -46 + 18 + 28 = 0$

$M_{12,12} = 1(-1) - 3\cdot 2 = -7 \neq 0$

$M_{123,124} = \begin{vmatrix} 1 & 3 & 4 \\ 2 & -1 & 3 \\ 4 & 5 & 10 \end{vmatrix} = 1\cdot(-10-15) - 3\cdot(20-12) + 4(10+4)$
$= -25 - 24 + 56 = 7 \neq 0$

$\text{rk}\,A = \underline{3}$

c) $A = \begin{pmatrix} 3 & 0 & 0 & 7 \\ 0 & 5 & 5 & 0 \\ 0 & 5 & 5 & 0 \end{pmatrix}$

not pivot pos.   not pivot pos.

$M_{123,123} = 3\cdot 0 = 0$

$M_{12,12} = 3\cdot 5 = 15 \neq 0$

$M_{123,124} = \begin{vmatrix} 3 & 0 & 7 \\ 0 & 5 & 0 \\ 0 & 5 & 0 \end{vmatrix} = 5\cdot 0 = 0$

$\text{rk}\,A = \underline{2}$

3. a) $A = \begin{pmatrix} 1 & 3 & t \\ 2 & 5 & 7 \end{pmatrix}$

$M_{12,12} = 5 - 6 = -1 \neq 0$

$\underline{\text{rk } A = 2}$

b) $A = \begin{pmatrix} a & 7 & -3 & 5 & 10 \\ 2 & -3 & 1 & 4 & 18 \\ 1 & 24 & -10 & 11 & 12 \end{pmatrix}$

$M_{123,123} = a(30-24) - 7(-20-1) - 3\cdot(48+3)$

$= 6a + 147 - 153 = \underline{6a - 6}$

$a \neq 1$: $M_{123,123} \neq 0 \Rightarrow \text{rk } A = \underline{3}$

$a = 1$: $M_{123,123} = 0$

$M_{12,12} = -3 - 14 = -17 \neq 0$

$M_{123,124} = \begin{vmatrix} 1 & 7 & 5 \\ 2 & -3 & 4 \\ 1 & 24 & 11 \end{vmatrix} = 1\cdot(-33-96)$

$-7(22-4) + 5(48+3) = -129 - 126 + 255 = \underline{0}$

$M_{123,125} = \begin{vmatrix} 1 & 7 & 10 \\ 2 & -3 & 18 \\ 1 & 24 & 12 \end{vmatrix} = 1(-36-432)$

$-7(24-18) + 10(48+3) = -468 - 42 + 510 = \underline{0}$

$a=1$: $A = \begin{pmatrix} 1 & 7 & -3 & 5 & 10 \\ 2 & -3 & 1 & 4 & 18 \\ 1 & 24 & -10 & 11 & 12 \end{pmatrix}$

not pivot pos. not pivot pos. not pivot pos.

$\text{rk}(A) = 2$ if $a = 1$

$= 3$ if $a \neq 1$

c) See next page.

4. a) $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & t & 13 \\ 1 & 3 & 9 & 24 \end{array}\right)$

$|A| = M_{123,123} = 1(18-3t) - 1(9-t) + 1\cdot(3-2)$

$= \underline{10 - 2t}$

$t \neq 5$: $|A| \neq 0 \Rightarrow$ <u>one solution</u>

$t = 5$: $M_{123,124} = \begin{vmatrix} 1 & 1 & 6 \\ 1 & 2 & 13 \\ 1 & 3 & 24 \end{vmatrix}$

$= 1(48-39) - 1(24-13) + 6(3-2)$

$= 9 - 11 + 6 = 4 \neq 0$

$M_{12,12} = 2 - 1 = 1 \neq 0$

$t = 5$: $\left(\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 5 & 13 \\ 1 & 3 & 9 & 24 \end{array}\right)$

<u>one solution</u> if $t \neq 5$

<u>no solutions</u> if $t = 5$

b) $(A|\underline{b}) = \left(\begin{array}{cccc|c} 1 & 4 & 5 & -3 & 6 \\ 2 & 7 & 1 & 0 & 4 \\ 1 & 5 & 4 & -8 & 1 \end{array}\right)$

$M_{123,123} = 1\cdot(28-5) - 4(8-1) + 5(10-7)$
$= 23 - 28 + 15 = 10 \neq 0$

<u>One free variable, inf. many solutions</u>

Std. Gauss: <u>w is free</u>

<u>Possible choices of free variable:</u>

$M_{123,123} \neq 0 \Rightarrow w$ free is possible

$M_{123,124} = \begin{vmatrix} 1 & 4 & -3 \\ 2 & 7 & 0 \\ 1 & 5 & -8 \end{vmatrix} = -3\cdot(10-7) \mp 8\cdot(7-8) = -9+8 = -1 \neq 0 \Rightarrow$ z free is possible

$M_{123,134} = \begin{vmatrix} 1 & 5 & -3 \\ 2 & 1 & 0 \\ 1 & 4 & 8 \end{vmatrix} = -3\cdot(8-1) \mp 8(1-10) = -27+72 = 45 \neq 0 \Rightarrow$ y free is possible

$M_{123,234} = \begin{vmatrix} 4 & 5 & -3 \\ 7 & 1 & 0 \\ 5 & 4 & -8 \end{vmatrix} = -3(28-5) - 8(4-35) = -69+248 \neq 0 \Rightarrow$ x free is possible

<u>Any choice of x, y, z, or w as free is possible.</u>

3. c) $A = \begin{pmatrix} 1 & a & b \\ a & b & 1 \end{pmatrix}$

$M_{12,12} = b - a^2$
$M_{12,13} = 1 - ab$
$M_{12,23} = a - b^2$

$\operatorname{rk} A = 2$, if $(a,b) \neq (1,1)$
$= 1$, if $(a,b) = (1,1)$

$\operatorname{rk} A < 2 \Leftrightarrow$ all 2-minors are zero

$b - a^2 = 0 \Rightarrow b = a^2$
$1 - ab = 0$
$a - b^2 = 0$

$a = b^2 = (a^2)^2 = a^4$
$a - a^4 = 0$
$a(1-a^3) = 0$
$a = 0$ or $a^3 = 1$
$b = a^2 = 0$ $\quad a = \sqrt[3]{1} = 1$
$a = 1$
$b = a^2 = 1$

<u>$1 - ab = 0$:</u>
$a=1, b=1$: ok
$a=0, b=0$: not poss.

<u>$a,b = (1,1)$:</u>

$A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$

$\operatorname{rk} A = 1$

$\boxed{\operatorname{rk} A < 2 \Leftrightarrow a = b = 1}$