GRA 6035 Mathematics

Problems for Lecture 4

Key problems

Problem 1.
Find all eigenvalues of A, and a base for the eigenspace E_λ for each eigenvalue λ, when A is the matrix:

$a)\ A = \begin{pmatrix} 3 & 7 \\ 7 & 3 \end{pmatrix} \quad b)\ A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \quad c)\ A = \begin{pmatrix} 2 & -4 \\ 3 & -1 \end{pmatrix} \quad d)\ A = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 4 \end{pmatrix} \quad e)\ A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad f)\ A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

Problem 2.
Determine whether the matrix A is diagonalizable, and find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ when this is possible:

$a)\ A = \begin{pmatrix} 3 & 7 \\ 7 & 3 \end{pmatrix} \quad b)\ A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \quad c)\ A = \begin{pmatrix} 2 & -4 \\ 3 & -1 \end{pmatrix} \quad d)\ A = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 4 \end{pmatrix} \quad e)\ A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix} \quad f)\ A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

Problem 3.
Find the eigenvalues of A, and show that A is diagonalizable:

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Problems from the Digital Workbook

Exercise problems	4.1 - 4.9 (full solutions in the workbook)
Exam problems	4.10 - 4.12 (full solutions in the workbook)

Answers to key problems

Problem 1.
a) Eigenvalues $\lambda_1 = -4$, $\lambda_2 = 10$ and eigenvectors $E_{-4} = \text{span}(\mathbf{v}_1)$ and $E_{10} = \text{span}(\mathbf{v}_2)$, where $\mathbf{v}_1 = \begin{pmatrix} -1 & 1 \end{pmatrix}^T$ and $\mathbf{v}_2 = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$
b) Eigenvalues $\lambda_1 = \lambda_2 = 2$ and eigenvectors $E_2 = \text{span}(\mathbf{v}_1)$, where $\mathbf{v}_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$
c) No eigenvalues or eigenvectors
d) Eigenvalues $\lambda_1 = \lambda_2 = 5$, $\lambda_3 = 3$ and eigenvectors $E_5 = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$ and $E_3 = \text{span}(\mathbf{v}_3)$, where $\mathbf{v}_1 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$, $\mathbf{v}_2 = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T$ and $\mathbf{v}_3 = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}^T$
e) Eigenvalues $\lambda_1 = \lambda_2 = 1$, $\lambda_3 = 4$ and eigenvectors $E_1 = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$ and $E_3 = \text{span}(\mathbf{v}_3)$, where $\mathbf{v}_1 = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix}^T$, $\mathbf{v}_2 = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}^T$ and $\mathbf{v}_3 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$
f) Eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 2$ and eigenvectors $E_2 = \text{span}(\mathbf{v}_1)$, where $\mathbf{v}_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$

Problem 2.
a) Yes, with $P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} -4 & 0 \\ 0 & 10 \end{pmatrix}$ b) No c) No

d) Yes, with $P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ e) Yes, with $P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ f) No

Problem 3.
The eigenvalues of A are $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 2$ and $\lambda_4 = -2$.

Solutions: Key problems Lecture 4

1. a) $A = \begin{pmatrix} 3 & 7 \\ 7 & 3 \end{pmatrix}$

Eigenval: $\begin{vmatrix} 3-\lambda & 7 \\ 7 & 3-\lambda \end{vmatrix} = 0$

$\lambda^2 - 6\lambda - 40 = 0$

$\lambda_1 = -4, \lambda_2 = 10$

Eigenvect: $\lambda = 10$: $\begin{pmatrix} -7 & 7 \\ 7 & -7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$-7x + 7y = 0$

y free $x = y$

$\underline{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \leftarrow \underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$E_{10} = \text{span}(\underline{v_1}), \ v_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$\lambda = -4$: $\begin{pmatrix} 7 & 7 \\ 7 & 7 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$7x + 7y = 0$ $\quad x = -y$, y = free

$v = \begin{pmatrix} -y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$E_{-4} = \text{span}(\underline{v_2}), \ v_2 = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

b) $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$

Eigenval: $\begin{vmatrix} 1-\lambda & 1 \\ -1 & 3-\lambda \end{vmatrix} = 0$ $\quad \lambda^2 - 4\lambda + 4 = 0$ $\quad \lambda_1 = \lambda_2 = 2$ (mult. 2)

Eigenvect:

$\lambda = 2$: $\begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad -x + y = 0$, y free, $x = y$ $\quad \underline{v} = \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} = y \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

$E_2 = \text{span}(\underline{v_1}), \ \underline{v_1} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

c) $A = \begin{pmatrix} 2 & -4 \\ 3 & -1 \end{pmatrix}$

Eigenval.: $\begin{vmatrix} 2-\lambda & -4 \\ 3 & -1-\lambda \end{vmatrix} = \lambda^2 - \lambda + 10 = 0$

$\lambda = \dfrac{1 \pm \sqrt{1-40}}{2}$ $\quad$ no (real) eigenvalues

Eigenvect: No (real) eigenvectors

d) $A = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 4 \end{pmatrix}$

Eigenval: $\begin{vmatrix} 4-\lambda & 0 & 1 \\ 0 & 5-\lambda & 0 \\ 1 & 0 & 4-\lambda \end{vmatrix} = 0$

$(5-\lambda)\cdot\left((4-\lambda)^2 - 1\right) = 0$

$(5-\lambda)(\lambda^2 - 8\lambda + 15) = 0$

$\lambda = 5$ or $\lambda = 3$ or $\lambda = 5$

$\lambda_1 = \lambda_2 = 5$ (mult. 2), $\lambda_3 = 3$

Eigenvect:

$\lambda = 5$: $\begin{pmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$-x + z = 0$
y, z free

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} z \\ y \\ z \end{pmatrix} = y\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$E_5 = \text{span}(\underline{v_1}, \underline{v_2})$, with $\underline{v_1} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, $\underline{v_2} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$

$\lambda = 3$: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$x + z = 0$
$2y = 0$
z free

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -z \\ 0 \\ z \end{pmatrix} = z\cdot\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

$E_3 = \text{span}(\underline{v_3})$, $\underline{v_3} = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$

e) See next page.

f) $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

Eigenval: $\lambda_1 = \lambda_2 = \lambda_3 = 2$
(since A is upper triangular)

alt: $\begin{vmatrix} 2-\lambda & 1 & 1 \\ 0 & 2-\lambda & 1 \\ 0 & 0 & 2-\lambda \end{vmatrix} = 0$

$(2-\lambda)(2-\lambda)(2-\lambda) = 0$

$\lambda_1 = \lambda_2 = \lambda_3 = 2$

Eigenvect:

$\lambda = 2$: $\begin{pmatrix} 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

$y = 0$
$\Uparrow$
$y + z = 0$
$z = 0$
x is free

$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x \\ 0 \\ 0 \end{pmatrix} = x\cdot\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

$E_2 = \text{span}(\underline{v_1})$, $\underline{v_1} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$

e) $A=\begin{pmatrix}2&1&1\\1&2&1\\1&1&2\end{pmatrix}$

Eigenval: $\begin{vmatrix}2-\lambda&1&1\\1&2-\lambda&1\\1&1&2-\lambda\end{vmatrix}=0$

$(2-\lambda)\cdot\left((2-\lambda)^2-1\right)-1\cdot(2-\lambda-1)+1\left(1-(2-\lambda)\right)=0$

$(2-\lambda)\left((2-\lambda)^2-1\right)+2\lambda-2=0$

$(2-\lambda)(\lambda^2-4\lambda+3)+2(\lambda-1)=0$

$(2-\lambda)(\lambda-3)(\lambda-1)+2(\lambda-1)=0$

$(\lambda-1)\cdot\left[(2-\lambda)(\lambda-3)+2\right]=0$

$\lambda=1$ or $-\lambda^2+5\lambda-4=0$

$\lambda=1,\ \lambda=4$

$\lambda_1=\lambda_2=1$ (mult. 2), $\lambda_3=4$

Eigenvect:

$\lambda=1$: $\begin{pmatrix}1&1&1\\1&1&1\\1&1&1\end{pmatrix}\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}$

$x+y+z=0$

y,z free

$\begin{pmatrix}x\\y\\z\end{pmatrix}=\begin{pmatrix}-y-z\\y\\z\end{pmatrix}=y\begin{pmatrix}-1\\1\\0\end{pmatrix}+z\begin{pmatrix}-1\\0\\1\end{pmatrix}$

$E_1=\text{span}(\underline{v_1},\underline{v_2})$, $\underline{v_1}=\begin{pmatrix}-1\\1\\0\end{pmatrix}$, $\underline{v_2}=\begin{pmatrix}-1\\0\\1\end{pmatrix}$

$\lambda=4$: $\left(\begin{array}{ccc|c}-2&1&1&0\\1&-2&1&0\\1&1&-2&0\end{array}\right)\to\left(\begin{array}{ccc|c}1&1&-2&0\\1&-2&1&0\\-2&1&1&0\end{array}\right)\begin{matrix}\\ \cdot(-1)\\ \cdot 2\end{matrix}\to\left(\begin{array}{ccc|c}1&1&-2&0\\0&-3&3&0\\0&3&-3&0\end{array}\right)$

$x+y-2z=0 \qquad x=-z+2z=z$

$-3y+3z=0 \qquad y=z$

z free

$\begin{pmatrix}x\\y\\z\end{pmatrix}=z\begin{pmatrix}1\\1\\1\end{pmatrix}$

$E_4=\text{span}(\underline{v_3})$, $\underline{v_3}=\begin{pmatrix}1\\1\\1\end{pmatrix}$

$D=\begin{pmatrix}1&0\\0&-4\end{pmatrix}$

2. a) Yes $P=\begin{pmatrix}1&-1\\1&1\end{pmatrix}$ ← see 1.

b) No (not enough eigenvectors)

c) No (not enough eigenvalues)

d) Yes $P=\begin{pmatrix}0&1&-1\\1&0&0\\0&1&1\end{pmatrix}$ $D=\begin{pmatrix}5&0&0\\0&5&0\\0&0&3\end{pmatrix}$

f) No (not enough eigenvectors)

e) Yes $P=\begin{pmatrix}-1&-1&1\\1&0&1\\0&1&1\end{pmatrix}$ $D=\begin{pmatrix}1&0&0\\0&1&0\\0&0&4\end{pmatrix}$

3. $A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$

A is diagonalizable since it is symmetric.

Eigenval:

$$\begin{vmatrix} -\lambda & 1 & 1 & 0 \\ 1 & -\lambda & 0 & 1 \\ 1 & 0 & -\lambda & 1 \\ 0 & 1 & 1 & -\lambda \end{vmatrix} = 0$$

$$-\lambda \cdot \begin{vmatrix} -\lambda & 0 & 1 \\ 0 & -\lambda & 1 \\ 1 & 1 & -\lambda \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 0 & 1 \\ 1 & -\lambda & 1 \\ 0 & 1 & -\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & -\lambda & 1 \\ 1 & 0 & 1 \\ 0 & 1 & -\lambda \end{vmatrix} = 0$$

$$-\lambda\left(-\lambda(\lambda^2-1) + 1\cdot\lambda\right) - \left(1\cdot(\lambda^2-1) + 1\cdot 1\right) + 1\cdot\left(1\cdot(-1) - 1\cdot(\lambda^2-1)\right) = 0$$

$$(\lambda^2-1)\cdot(\lambda^2-1-1) - \lambda^2 - 1 - 1 = 0$$

$$(\lambda^2-1)(\lambda^2-2) - \lambda^2 - 2 = 0$$

$$\lambda^4 - \lambda^2 - 2\lambda^2 + \cancel{2} - \lambda^2 - \cancel{2} = 0$$

$$\lambda^4 - 4\lambda^2 = 0 \qquad \lambda^2(\lambda^2-4) = 0$$

$$\lambda^2(\lambda-2)(\lambda+2) = 0$$

$$\underline{\lambda_1 = \lambda_2 = 0 \text{ (mult. 2)}, \ \lambda_3 = 2, \ \lambda_4 = -2}$$

Alternative computations of eigenvalues:

* $\dim E_0 = 2$ since $E_0: A\cdot \underline{x} = \underline{0}$ $(\lambda = 0)$ and $\dim E_0 = 4 - rk(A) = 4 - 2 = 2$ (See that $rk A = 2$ by inspection)

  A symmetric $\Rightarrow$ A diagonalizable $\Rightarrow$ mult.$(\lambda=0) = \dim E_0 = 2$

  Hence $\underline{\lambda_1 = \lambda_2 = 0}$

* $\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 = tr(A) = 0 \Rightarrow \lambda_3 + \lambda_4 = 0 \Rightarrow \underline{\lambda_4 = -\lambda_3}$

* $A \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \Rightarrow \lambda = 2$ is an eigenvalue $\Rightarrow \underline{\lambda_3 = 2}, \underline{\lambda_4 = -2}$