GRA 6035 Mathematics

Problems for Lecture 4

Key problems

Problem 1.
Find all eigenvalues of A, and a base for the eigenspace E_λ for each eigenvalue λ, when A is the matrix:

a) $A = \begin{pmatrix} 3 & 7 \\ 7 & 3 \end{pmatrix}$ *b)* $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$ *c)* $A = \begin{pmatrix} 2 & -4 \\ 3 & -1 \end{pmatrix}$ *d)* $A = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 4 \end{pmatrix}$ *e)* $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ *f)* $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

Problem 2.
Determine whether the matrix A is diagonalizable, and find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ when this is possible:

a) $A = \begin{pmatrix} 3 & 7 \\ 7 & 3 \end{pmatrix}$ *b)* $A = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix}$ *c)* $A = \begin{pmatrix} 2 & -4 \\ 3 & -1 \end{pmatrix}$ *d)* $A = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 4 \end{pmatrix}$ *e)* $A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$ *f)* $A = \begin{pmatrix} 2 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$

Problem 3.
Find the eigenvalues of A, and show that A is diagonalizable:

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

Problems from the Digital Workbook

Exercise problems	4.1 - 4.9 (full solutions in the workbook)
Exam problems	4.10 - 4.12 (full solutions in the workbook)

Answers to key problems

Problem 1.
a) Eigenvalues $\lambda_1 = -4,\ \lambda_2 = 10$ and eigenvectors $E_{-4} = \text{span}(\mathbf{v}_1)$ and $E_{10} = \text{span}(\mathbf{v}_2)$, where $\mathbf{v}_1 = \begin{pmatrix} -1 & 1 \end{pmatrix}^T$ and $\mathbf{v}_2 = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$
b) Eigenvalues $\lambda_1 = \lambda_2 = 2$ and eigenvectors $E_2 = \text{span}(\mathbf{v}_1)$, where $\mathbf{v}_1 = \begin{pmatrix} 1 & 1 \end{pmatrix}^T$
c) No eigenvalues or eigenvectors
d) Eigenvalues $\lambda_1 = \lambda_2 = 5,\ \lambda_3 = 3$ and eigenvectors $E_5 = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$ and $E_3 = \text{span}(\mathbf{v}_3)$, where $\mathbf{v}_1 = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T$, $\mathbf{v}_2 = \begin{pmatrix} 1 & 0 & 1 \end{pmatrix}^T$ and $\mathbf{v}_3 = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}^T$
e) Eigenvalues $\lambda_1 = \lambda_2 = 1,\ \lambda_3 = 4$ and eigenvectors $E_1 = \text{span}(\mathbf{v}_1, \mathbf{v}_2)$ and $E_3 = \text{span}(\mathbf{v}_3)$, where $\mathbf{v}_1 = \begin{pmatrix} -1 & 1 & 0 \end{pmatrix}^T$, $\mathbf{v}_2 = \begin{pmatrix} -1 & 0 & 1 \end{pmatrix}^T$ and $\mathbf{v}_3 = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^T$
f) Eigenvalues $\lambda_1 = \lambda_2 = \lambda_3 = 2$ and eigenvectors $E_2 = \text{span}(\mathbf{v}_1)$, where $\mathbf{v}_1 = \begin{pmatrix} 1 & 0 & 0 \end{pmatrix}^T$

Problem 2.
a) Yes, with $P = \begin{pmatrix} -1 & 1 \\ 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} -4 & 0 \\ 0 & 10 \end{pmatrix}$ b) No c) No

d) Yes, with $P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ e) Yes, with $P = \begin{pmatrix} -1 & -1 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \end{pmatrix}$, $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{pmatrix}$ f) No

Problem 3.
The eigenvalues of A are $\lambda_1 = \lambda_2 = 0$, $\lambda_3 = 2$ and $\lambda_4 = -2$.