GRA 6035 Mathematics

Problems for Lecture 5

Key problems

Problem 1.
Find the equilibrium state $\mathbf{v}$ of the Markov chains with transition matrix A:

a) $A = \begin{pmatrix} 0.30 & 0.15 \\ 0.70 & 0.85 \end{pmatrix}$ *b)* $A = \begin{pmatrix} 0.86 & 0.42 \\ 0.14 & 0.58 \end{pmatrix}$ *c)* $A = \begin{pmatrix} 0.75 & 0.02 & 0.10 \\ 0.20 & 0.90 & 0.20 \\ 0.05 & 0.08 & 0.70 \end{pmatrix}$ *d)* $A = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

Problem 2.
Determine the definiteness of the symmetric matrix:

a) $A = \begin{pmatrix} 7 & 4 \\ 4 & 3 \end{pmatrix}$ *b)* $A = \begin{pmatrix} -1 & 1 \\ 1 & -3 \end{pmatrix}$ *c)* $A = \begin{pmatrix} 4 & 0 & 1 \\ 0 & 5 & 0 \\ 1 & 0 & 4 \end{pmatrix}$ *d)* $A = \begin{pmatrix} 2 & 3 & -5 \\ 3 & 7 & 0 \\ -5 & 0 & 35 \end{pmatrix}$ *e)* $A = \begin{pmatrix} -1 & -2 & -2 \\ -2 & -4 & -4 \\ -2 & -4 & -2 \end{pmatrix}$

Problem 3.
Find the symmetric matrix of the quadratic form, and determine its definiteness:

a) $f(x,y) = x^2 - 8xy + 3y^2$ *b)* $f(x,y,z) = 2x^2 - 2xz + 3y^2 + z^2$ *c)* $f(x,y,z) = 3x^2 + 4xy - 4xz + 3y^2 + 4yz + 8z^2$
d) $f(x,y) = 2xy - y^2$ *e)* $f(x,y,z,w) = xw - yz$

Problem 4.
Determine the definiteness of the symmetric matrix:

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & -1 \end{pmatrix}$$

Problems from the Digital Workbook

Exercise problems	5.1 - 5.9 (full solutions in the workbook)
Exam problems	5.10 - 4.15 (full solutions in the workbook)
	Midterm exam 01/2018 Question 1-6, 8
	Midterm exam 05/2018 Question 1-6, 8

Answers to key problems

Problem 1.
a) $\mathbf{v} = (3/17\ 14/17)^T$ b) $\mathbf{v} = (3/4\ 1/4)^T$ c) $\mathbf{v} = (2/15\ 10/15\ 3/15)^T$ d) No equilibrium unless $\mathbf{v}_0 = (a\ b\ a)^T$

Problem 2.
a) Positive definite b) Negative definite c) Positive definite d) Positive semi-definite e) Indefinite

Problem 3.
a) Indefinite b) Positive definite c) Positive semi-definite d) Indefinite e) Indefinite

Problem 4.
Positive semidefinite

LaTeX TikZposter

Solutions: Key problems for Lecture 5

1.

a) $A=\begin{pmatrix}0.30 & 0.15\\ 0.70 & 0.85\end{pmatrix}$

regular Markov chain since all $a_{ij}>0$ $\Rightarrow$

$\underline{\lambda=1}$: $\begin{pmatrix}-0.70 & 0.15\\ 0.70 & -0.15\end{pmatrix}\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0\\ 0\end{pmatrix}$

$-0.7x+0.15y=0$

y free, $x=0.15/0.70\, y$

$\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0.15/0.70\, y\\ y\end{pmatrix}=y\begin{pmatrix}0.15/0.7\\ 1\end{pmatrix}$

$\underline{x+y=1}$: $y\cdot {}^{0.15}/_{0.7}+y=1$

$y\cdot\left({}^{0.85}/_{0.7}\right)=1$

$y={}^{0.7}/_{0.85}$

$\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}0.15/0.85\\ 0.70/0.85\end{pmatrix}=\begin{pmatrix}15/85\\ 70/85\end{pmatrix}$

$$\underline{v}=\underline{\underline{\begin{pmatrix}3/17\\ 14/17\end{pmatrix}}}$$

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b) $A=\begin{pmatrix}0.86 & 0.42\\ 0.14 & 0.58\end{pmatrix}$

regular

$\begin{pmatrix}-0.14 & 0.42\\ 0.14 & -0.42\end{pmatrix}\to\begin{pmatrix}-14 & 42\\ 0 & 0\end{pmatrix}$

$-14x+42y=0$

$\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}42\\ 14\end{pmatrix}t$

$x+y=1$: $42t+14t=56t=1$

$t={}^{1}/_{56}$

$$\underline{v}=\begin{pmatrix}x\\ y\end{pmatrix}=\begin{pmatrix}42/56\\ 14/56\end{pmatrix}=\underline{\underline{\begin{pmatrix}3/4\\ 1/4\end{pmatrix}}}$$

c) $A=\begin{pmatrix}0.75 & 0.02 & 0.10\\ 0.20 & 0.90 & 0.20\\ 0.05 & 0.08 & 0.70\end{pmatrix}$

regular

$\underline{\lambda=1}$: $\begin{pmatrix}-0.25 & 0.02 & 0.10\\ 0.20 & -0.10 & 0.20\\ 0.05 & 0.08 & -0.30\end{pmatrix}$ $\cdot 100$ $\cdot 100$ $\cdot 100$

$\to\begin{pmatrix}5 & 8 & -30\\ 20 & -10 & 20\\ -25 & 2 & 10\end{pmatrix}\begin{matrix}\\ -4\\ 5\end{matrix}\to\begin{pmatrix}5 & 8 & -30\\ 0 & -42 & 140\\ 0 & 42 & -140\end{pmatrix}\to\begin{pmatrix}5 & 8 & -30\\ 0 & -42 & 140\\ 0 & 0 & 0\end{pmatrix}$

$-42y+140z=0$

$\underline{y=\frac{140}{42}z=\frac{10}{3}z}$

$5x+8y-30z=0$

$5x=-8\left(\frac{140}{42}z\right)+30z=\frac{30\cdot 42-8\cdot 140}{42}z$

$=\frac{1260-1120}{42}z=\underline{\frac{140}{42}z}=\frac{10}{3}z$

$\underline{x=\frac{1}{5}\cdot\frac{10}{3}z=\frac{2}{3}z}$

$x+y+z=\frac{140}{42}z+\frac{28}{42}z+z$

$=\frac{2}{3}z+\frac{10}{3}z+\frac{3}{3}z=1$

$5z=1\Rightarrow z={}^{1}/_{5}$

$$\underline{v}=\begin{pmatrix}x\\ y\\ z\end{pmatrix}=\underline{\underline{\begin{pmatrix}2/15\\ 10/15\\ 3/15\end{pmatrix}}}$$

d) $A=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}$

not regular

Compute: A^n:

$$A^2=\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}\cdot\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}=\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$$

$$\Rightarrow A^n=\begin{cases}\begin{pmatrix}0&0&1\\0&1&0\\1&0&0\end{pmatrix}, & n \text{ odd}\\ \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}, & n \text{ even}\end{cases}$$

$$\underline{v}_0=\begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix} \Rightarrow A^n\cdot\underline{v}_0=v_n=\begin{cases}A^n\cdot v_0=\begin{pmatrix}z_0\\y_0\\x_0\end{pmatrix}, & n \text{ odd}\\ A^n\cdot v_0=\begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix}, & n \text{ even}\end{cases}$$

$x_0=z_0$: $\underline{v}=\lim_{n\to\infty}A^n\cdot\underline{v}_0=\begin{pmatrix}x_0\\y_0\\x_0\end{pmatrix}$ eq. state

$x_0\neq z_0$: no equilibrium state

$$\begin{pmatrix}x_0\\y_0\\z_0\end{pmatrix}\rightleftarrows\begin{pmatrix}z_0\\y_0\\x_0\end{pmatrix}$$ no convergence as $n\to\infty$

2. a) $A=\begin{pmatrix}7&4\\4&3\end{pmatrix}$ $D_1=7>0$, $D_2=21-16=5>0$ A pos. definite

b) $A=\begin{pmatrix}-1&1\\1&-3\end{pmatrix}$ $D_1=-1<0$, $D_2=3-1=2>0$ A neg. definite

c) $A=\begin{pmatrix}4&0&1\\0&5&0\\1&0&4\end{pmatrix}$ $D_1=4>0$, $D_2=20>0$, $D_3=5\cdot(16-1)>0$ A pos. def.

d) $A=\begin{pmatrix}2&3&-5\\3&7&0\\-5&0&35\end{pmatrix}$ $D_1=2>0$, $D_2=14-9=5>0$, $D_3=-5\cdot(0+35)+35\cdot 5=0$

$\Delta_1=2,7,35\geq 0$
$\Delta_2=5,245,45\geq 0$
$\Delta_3=0\geq 0$

A pos. semidef.

e) $A=\begin{pmatrix}-1&-2&-2\\-2&-4&-4\\-2&-4&-2\end{pmatrix}$

$D_1=-1$ $\quad\Delta_1=-1,-4,-2\leq 0$
$D_2=0$ $\quad\Delta_2=0,-2$
$D_3=-2\cdot 0+4\cdot 0-2\cdot 0=0$

A indefinite

3. a) $A=\begin{pmatrix}1 & -4\\ -4 & 3\end{pmatrix}$ $\quad D_1=1$, $D_2=3-16=-13$ Indefinite

b) $A=\begin{pmatrix}2 & 0 & -1\\ 0 & 3 & 0\\ -1 & 0 & 1\end{pmatrix}$ $\quad D_1=2$, $D_2=6$, $D_3=3\cdot(2-1)=3$ pos. defin.

c) $A=\begin{pmatrix}3 & 2 & -2\\ 2 & 3 & 2\\ -2 & 2 & 8\end{pmatrix}$

$D_1=3$

$D_2=9-4=5$

$D_3=-2\cdot(10)-2\cdot 10+8\cdot 5$

$=-20-20+40=0$

$\Delta_1=3,3,8$

$\Delta_2=5,20,20$

$\Delta_3=0$

pos. semidefin

d) $A=\begin{pmatrix}0 & +1\\ +1 & -1\end{pmatrix}$ $\quad D_1=0$, $D_2=-1$ indefinite

e) $A=\begin{pmatrix}0 & 0 & 0 & 1\\ 0 & 0 & -1 & 0\\ 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\end{pmatrix}$

$D_1=0$

$D_2=0$

$D_3=0$

$\Delta_1=0,0,0,0$

$\Delta_2=0,0,-1 \Rightarrow$ indefinite

4. $A=\begin{pmatrix}1 & 0 & 0 & -1\\ 0 & 1 & -1 & 0\\ 0 & -1 & 1 & 0\\ -1 & 0 & 0 & +1\end{pmatrix}$

$D_1=1$ $\quad \Delta_1=1,1,1,1\geq 0$

$D_2=1$ $\quad \Delta_2=1,1,0,0,1,1\geq 0$

$D_3=0$ $\quad \Delta_3=0,0,0,0\geq 0$

$D_4=0$ $\quad \Delta_4=0\geq 0$

pos. semidefin