GRA 6035 Mathematics

Problems for Lecture 7

Key problems

Problem 1.
Check if the sets given by these conditions are convex and compact (closed and bounded). It can be useful to sketch the sets:
a) $x, y \geq 0$ and $2x + 3y \leq 6$ *b)* $4x^2 + 9y^2 \leq 36$ *c)* $x, y \geq 1$ and $2x + 3y \geq 12$ *d)* $xyz \leq 1$ and $x, y, z > 0$

Problem 2.
Determine whether the functions $f(x, y) = |x - y|$ and $g(x, y, z) = 1 - e^{x-y+z}$ are convex or concave.

Problem 3.
Solve the Lagrange problems. You may assume that all admissible points satisfy the NDCQ:
a) $\max f(x, y, z) = x + 2y + 3z$ when $2x^2 + y^2 + 2z^2 = 9$ *b)* $\max / \min f(x, y, z) = x^4 + y^4 + z^4$ when $2x^2 + y^2 + 2z^2 = 9$

Problem 4.
Solve the Kuhn-Tucker problems. You may assume that all admissible points satisfy the NDCQ:
a) $\max f(x, y, z) = x - 2y + z$ when $x^2 + y^2 + z^2 \leq 3$ *b)* $\max f(x, y, z) = \ln(xyz)$ when $2x^2 + y^2 + 2z^2 \leq 6$

Problems from the Digital Workbook

Exercise problems	7.1 - 7.10 (full solutions in the workbook)
Exam problems	7.11 (full solutions in the workbook)

Answers to key problems

Problem 1.
a) Convex and compact set b) Convex and compact set c) Convex but not compact set (not bounded) d) Not convex and not compact set (not bounded)

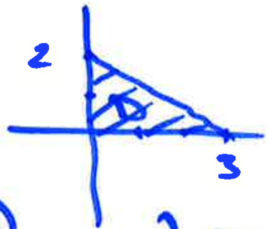
Problem 2.
The function f is convex, but not concave. The function g is concave, but not convex.

Problem 3.
a) $f_{\max} = 9$ b) $f_{\max} = 81$, $f_{\min} = 9$

Problem 4.
a) $f_{\max} = 3\sqrt{2}$ b) $f_{\max} = \ln(2)/2$

Solutions: Key problems Lecture 7

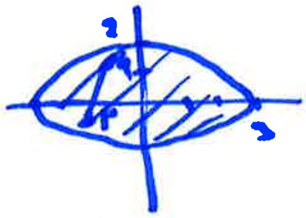
1. a) $x, y \geq 0$
$2x + 3y \leq 6$



Compact: closed: ✓ $(\leq, \geq)$
bounded: $0 \leq x \leq 3$
$0 \leq y \leq 2$ ✓
} D is <u>compact</u>

Convex: any pts P, Q in D
⇓ (see the sketch)
$[P, Q]$ in D
} D is <u>convex</u>

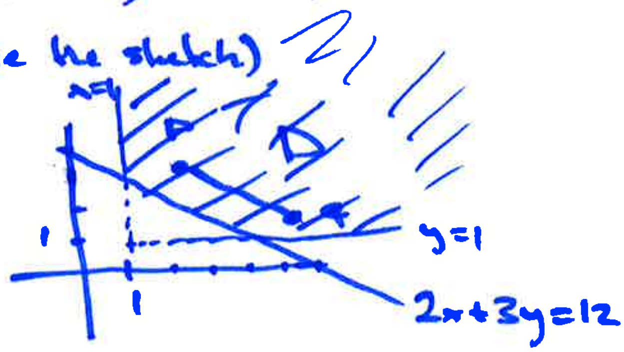
b) $4x^2 + 9y^2 \leq 36$:



$-3 \leq x \leq 3$
$-2 \leq y \leq 2$
} D is <u>compact</u> (bounded)

D is <u>convex</u> (see the sketch)

c) $x, y \geq 1$
$2x + 3y \geq 12$



<u>not compact</u>
(no upper bound on x or y)

<u>convex</u> (see the sketch)

d) $xyz \leq 1$
$x, y, z > 0$

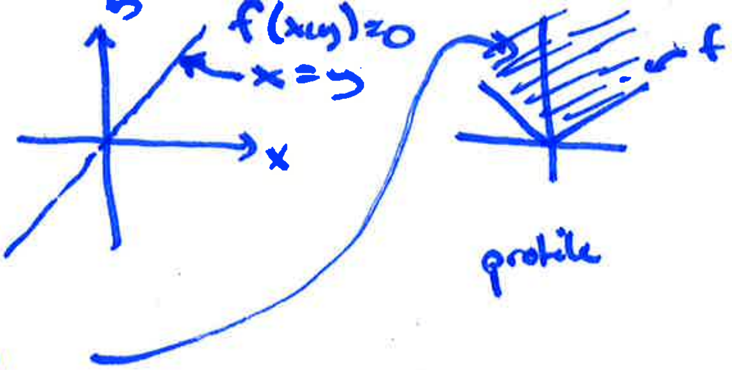
<u>not compact</u> (for example, $x = 1$, $y = t$, $z = 1/t$ for $t > 0$ is in D, $y = t \to \infty$ within D so D <u>not bounded</u>)

$z = 1$

<u>not convex</u>

$P = (2, 1/2, 1)$, $(1/2, 2, 1) = Q$ in D
midpt. $(5/4, 5/4, 1)$ not in D since $\frac{5}{4} \cdot \frac{5}{4} \cdot 1 > 1$

2. $f(x,y) = |x-y|$ not differentiable fn.



set of pts over graph of f
is <u>convex set</u> $\Rightarrow$ f <u>convex fn.</u> (not concave)

$g(x,y,z) = k - e^u$, $u = x-y+z$

$g'_x = -e^u \cdot 1$

$g'_y = -e^u \cdot (-1)$

$g'_z = -e^u \cdot 1$

$$H(g) = \begin{pmatrix} -e^u & e^u & -e^u \\ e^u & -e^u & e^u \\ -e^u & e^u & -e^u \end{pmatrix}$$

$D_1 = -e^u < 0$ for all x,y,z

$D_2 = D_3 = 0$ since rk $H(g) = 1$

<u>REC:</u> $H(g)$ neg. semidefin.

$\Downarrow$

g <u>concave</u> (not convex since $D_1 < 0$)

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3. a) max $f = x+2y+3z$ when $2x^2+y^2+2z^2=9$

$L = x+2y+3z - \lambda(2x^2+y^2+2z^2)$

$L'_x = 1 - \lambda \cdot 4x = 0$ $\quad x = 1/4\lambda \quad (\lambda \neq 0)$

$L'_y = 2 - \lambda \cdot 2y = 0$ $\quad y = 2/2\lambda = 4/4\lambda$

$L'_z = 3 - \lambda \cdot 4z = 0$ $\quad z = 3/4\lambda$

$2x^2+y^2+2z^2=9$

$$2(1/4\lambda)^2 + (4/4\lambda)^2 + 2(3/4\lambda)^2 = 9$$

$$\frac{2 \cdot 1^2 + 4^2 + 2 \cdot 3^2}{(4\lambda)^2} = \frac{36}{(4\lambda)^2} = 9$$

$(4\lambda)^2 = \frac{36}{9} = 4 \quad 4\lambda = \pm 2 \quad \underline{\lambda = \pm 1/2}$

<u>Cand:</u>

$(x,y,z;\lambda) = \underline{(1/2, 2, 3/2; 1/2)}$ $f = 9$

$\underline{(-1/2, -2, -3/2; -1/2)}$ $f = -9$

1) D compact (bounded)

$\Downarrow$ EVT

there is a max

2) NDCQ satisfied

$\Rightarrow$ Max is last cand. pt: $f_{max} = 9$ at $(x,y,z) = (1/2, 2, 3/2)$

b) max/min $f = x^4 + y^4 + z^4$ when $2x^2 + y^2 + 2z^2 = 9$

$h' = x^4 + y^4 + z^4 - \lambda(2x^2 + y^2 + 2z^2)$

$h'_x = 4x^3 - \lambda \cdot 4x = 0 \qquad 4x(x^2 - \lambda) = 0$

$h'_y = 4y^3 - \lambda \cdot 2y = 0 \qquad 2y(2y^2 - \lambda) = 0$

$h'_z = 4z^3 - \lambda \cdot 4z = 0 \qquad 4z(z^2 - \lambda) = 0$

$\boxed{2x^2 + y^2 + 2z^2 = 9}$

$\boxed{\begin{array}{l} x = 0 \text{ or } x^2 = \lambda \\ y = 0 \text{ or } y^2 = \lambda/2 \\ z = 0 \text{ or } z^2 = \lambda \end{array}}$

a) $x = y = z = 0$: no solution in C

b1) $x = y = 0,\ z^2 = \lambda$: $2z^2 = 9 \Rightarrow z^2 = 9/2$ $\quad (x,y,z) = (0, 0, \pm\sqrt{9/2}; 9/2)$ $\quad f = 81/4 = 20{,}25$

b2) $x = z = 0,\ y^2 = \lambda/2$: $y^2 = 9 \Rightarrow y = \pm 3$ $\quad (0, \pm 3, 0; 18)$ $\quad f = 81$

b3) $y = z = 0,\ x^2 = \lambda$: same as b1) by symmetry $\quad (\pm\sqrt{9/2}, 0, 0; 9/2)$ $\quad f = 20.25$

c1) $x = 0,\ y^2 = \lambda/2,\ z^2 = \lambda$: $\frac{\lambda}{2} + 2\lambda = 9$, $\frac{5}{2}\lambda = 9$, $\lambda = 18/5$ $\quad y^2 = 18/10 = 9/5$, $z^2 = 18/5$ $\quad (0, \pm\sqrt{9/5}, \pm\sqrt{18/5}; \frac{18}{5})$

$f = (9/5)^2 + (18/5)^2 = \frac{81 + 324}{25} = 16{,}2$

c2) $x^2 = \lambda,\ y = 0,\ z^2 = \lambda$: $2\lambda + 2\lambda = 9$, $4\lambda = 9$, $\lambda = 9/4$ $\quad (\pm 3/2, 0, \pm 3/2)$ $\quad \lambda = 9/4$ $\quad f = 10{,}125$

c3) $x^2 = \lambda,\ y^2 = \lambda/2,\ z = 0$: same as c1 $\quad (\pm\sqrt{18/5}, \pm\sqrt{9/5}, 0)$ $\quad \lambda = \frac{18}{5}$ $\quad f = 16.2$

d) $x^2 = \lambda,\ y^2 = \lambda/2,\ z^2 = \lambda$: $2\lambda + \frac{\lambda}{2} + 2\lambda = 9$, $\frac{9}{2}\lambda = 9$, $\lambda = 2$ $\quad (\pm\sqrt{2}, \pm 1, \pm\sqrt{2})$ $\quad \lambda = 2$ $\quad f = 9$

D compact (bounded) $\underset{EVT}{\Rightarrow}$ there is max/min

NDCQ satisfied

$\Downarrow$

max/min = best cand pts :

Min: $f_{min} = 9$ at $(\pm\sqrt{2}, 1, \pm\sqrt{2})$

Max: $f_{max} = 81$ at $(0, \pm 3, 0)$

4. a) max $f = x - 2y + z$ when $x^2+y^2+z^2 \leq 3$

$h = x - 2y + z - \lambda(x^2+y^2+z^2)$ (pb. in std form: max, $\leq$)

FOC
$$\begin{cases} h'_x = 1 - \lambda \cdot 2x = 0 \\ h'_y = -2 - \lambda \cdot 2y = 0 \\ h'_z = 1 - \lambda \cdot 2z = 0 \end{cases}$$

C : $x^2+y^2+z^2 \leq 3$

CSC: $\lambda \geq 0$, $\lambda(x^2+y^2+z^2-3) = 0$

a) $x^2+y^2+z^2 < 3$: $\lambda = 0$

$L'_x = 1 - \lambda \cdot 2x = 0$ impossible

$\Downarrow$

no cand. pts in a)

b) $x^2+y^2+z^2 = 3$: $\lambda \geq 0$

$x = \frac{1}{2\lambda}$ $(\lambda \neq 0)$

$y = \frac{-2}{2\lambda}$

$z = \frac{1}{2\lambda}$

$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{-2}{2\lambda}\right)^2 + \left(\frac{1}{2\lambda}\right)^2 = \frac{1+4+1}{(2\lambda)^2} = 3$

$\frac{6}{(2\lambda)^2} = 3$ $\quad (2\lambda)^2 = \frac{6}{3} = 2$

$2\lambda = \pm\sqrt{2}$

$\lambda = \pm\sqrt{2}/2$

$\lambda \geq 0 \Rightarrow \lambda = \sqrt{2}/2$

cand. pt: $\rightarrow$ $x = 1/\sqrt{2}$ $\quad y = -2/\sqrt{2}$ $\quad z = 1/\sqrt{2}$

$f = \frac{1}{\sqrt{2}} - 2\left(\frac{-2}{\sqrt{2}}\right) + \frac{1}{\sqrt{2}}$

$= \frac{6}{\sqrt{2}} = \frac{6 \cdot \sqrt{2}}{\sqrt{2} \cdot \sqrt{2}} = \frac{6\sqrt{2}}{2} = 3\sqrt{2}$

$x^2+y^2+z^2 \leq 3$ bounded $\underset{EVT}{\Rightarrow}$ max exists

NDCQ satisfied

$\Rightarrow$ max is attained at best cand. pt: $\quad f_{max} = \underline{\underline{3\sqrt{2}}}$ at

$$\underline{\underline{(x,y,z) = \left(1/\sqrt{2}, -2/\sqrt{2}, 1/\sqrt{2}\right)}}$$

$$\left(\tfrac{1}{2}\sqrt{2}, -\sqrt{2}, \tfrac{1}{2}\sqrt{2}\right)$$

b) max $\ln(xyz)$ when $2x^2+y^2+2z^2 \le 6$

<u>Note:</u> $\ln(xyz)$ is defined when $xyz > 0$
not all pts in $\boxed{D: 2x^2+y^2+2z^2 \le 6}$ satisfy $xyz > 0$

Therefore $xyz > 0$ is an <u>additional condition</u>.

We solve the problem as an ordinary KT-problem, with $2x^2+y^2+2z^2 \le 6$ as constraint, and check that candidate pts satisfy $\underline{xyz>0}$ in addition.

$$L = \ln(xyz) - \lambda(2x^2+y^2+2z^2)$$

$$L'_x = \frac{1}{xyz}\cdot yz - \lambda \cdot 4x = 0 \qquad \frac{1}{x} - 4\lambda x = 0 \qquad 1 = 4\lambda \cdot x^2$$

$$L'_y = \frac{1}{xyz}\cdot xz - \lambda \cdot 2y = 0 \qquad \frac{1}{y} - 2\lambda y = 0 \qquad 1 = 2\lambda \cdot y^2$$

$$L'_z = \frac{1}{xyz}\cdot xy - \lambda \cdot 4z = 0 \qquad \frac{1}{z} - 4\lambda z = 0 \qquad 1 = 4\lambda \cdot z^2$$

a) $\underline{2x^2+y^2+2z^2 < 6}$: $\lambda = 0$ $\quad$ <u>impossible</u> since $1 = 4\lambda \cdot x^2$
$\Rightarrow$ no cand. pts. in this case

b) $\underline{2x^2+y^2+2z^2 = 6}$: $\lambda \ge 0$

$$2x^2+y^2+2z^2 = \frac{2}{4\lambda} + \frac{2}{4\lambda} + \frac{2}{4\lambda} = 6$$

$$x^2 = \frac{1}{4\lambda} \quad y^2 = \frac{1}{2\lambda} = \frac{2}{4\lambda} \quad z^2 = \frac{1}{4\lambda} \Rightarrow \quad \frac{6}{4\lambda} = 6 \quad 4\lambda = 1 \quad \underline{\lambda = 1/4}$$

$\Rightarrow x^2 = \frac{1}{1} = 1 \qquad y^2 = \frac{2}{1} = 2 \qquad z^2 = \frac{1}{1} = 1$

$x = \pm 1 \qquad y = \pm\sqrt{2} \qquad z = \pm 1$

Cand. pts: $(x,y,z;\lambda) = (\pm 1, \pm\sqrt{2}, \pm 1; 1/4)$ $\quad \lambda = 1/4 \geq 0$ Vok.

Cand. pts with $xyz > 0$:

$(1,\sqrt{2},1)$, $(-1,-\sqrt{2},1)$, $(-1,\sqrt{2},-1)$, $(1,-\sqrt{2},-1)$

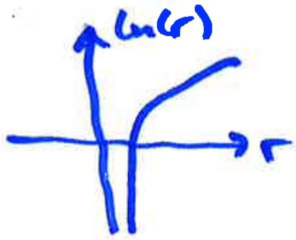
$f = \ln\sqrt{2}$ $\quad f = \ln(\sqrt{2})$ $\quad f = \ln(\sqrt{2})$ $\quad f = \ln(\sqrt{2})$

Conclusion:

If f was defined for all x,y,z in $D: 2x^2+y^2+2z^2 \leq 6$, which is bounded, then the pb. has a max because of EVT.

Since f is only defined for $xyz>0$, there could be pts with $xyz>0$, but close to zero, for which $f(x,y,z) \to \infty$.

Check: $xyz = r$, where $r>0$ is small, gives $f = \ln(xyz) = \ln(r)$, and $\ln r \to -\infty$ when $r \to 0^+$.



Therefore, the problem has a max even when we take into account that $xyz>0$ for f to be defined.

(However, $f \to -\infty$ inside D, so there would not be a min).

NDCQ satisfied $\Downarrow$ Best candidate pt. is max

$f_{max} = \ln(\sqrt{2}) = \ln(2^{1/2}) = \frac{1}{2}\ln 2$

is attained at

$(x,y,z) = (1,\sqrt{2},1), (-1,-\sqrt{2},1), (-1,\sqrt{2},-1), (1,-\sqrt{2},-1)$.