# GRA 6035 Mathematics

## Problems for Lecture 8

## Key problems

**Problem 1.**
Use the SOC to show that the given point is a solution of the constrained optimization problem:
*a)* $(x^*, y^*) = (1, 1)$ is a minimum for: $\min f(x, y) = x^2 + y^2$ when $xy = 1$
*b)* $(x^*, y^*, z^*) = (2, 0, 0)$ is a minimum for: $\min f(x, y, z) = x^2 + y^2 + z^2$ when $3x^2 + 2y^2 + 2z^2 \geq 12$

**Problem 2.**
Determine if there are any admissible points such that the NDCQ fails when the constraints are given by:
*a)* $xyz = 1$ *b)* $3x^2 + 3y^2 + 8z^2 \geq 1$ *c)* $x^3 + y^3 + z^3 = 0$ *d)* $xy - zw = 1$ and $x + y + z + w = 4$

**Problem 3.**
Solve the constrained optimization problems:
*a)* $\max f(x, y, z) = 2 - x^2 + 4xy - 5y^2 + 2yz - 2z^2$ when $3x - 2y + z = 10$
*b)* $\max f(x, y, z, w) = xz + yw$ when $x^2 + y^2 \leq 1$ and $4z^2 + 9w^2 \leq 36$

## Problems from the Digital Workbook

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|---|---|
| Exercise problems | 8.1 - 8.9 (full solutions in the workbook) |
| Exam problems | 8.10 - 8.13 (full solutions in the workbook) |

## Answers to key problems

**Problem 2.**
a) None b) None c) $(x, y, z) = (0, 0, 0)$ d) None

**Problem 3.**
a) $f_{\max} = 0$ b) $f_{\max} = 3$

# Solutions: Key problems – Lecture 8

**1.** a) $\min f(x,y) = x^2+y^2$ when $xy = 1$

$\mathcal{L} = x^2 + y^2 - \lambda(xy)$

$\mathcal{L}'_x = 2x - \lambda y = 0$

$\mathcal{L}'_y = 2y - \lambda x = 0$

$xy = 1$

$(x^*,y^*) = (1,1)$ gives $\lambda = \underline{2}$

$\Downarrow$

$\underline{(1,1;2)}$ ord. cond. pt.

$h(x,y) = \mathcal{L}(x,y;2) = x^2+y^2-2xy$

$h'_x = 2x - 2y \qquad H(h) = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix}$

$h'_y = -2x + 2y$

$D_1 = 2 > 0$

$D_2 = 0$

$\Downarrow$

$(x^*,y^*) = (1,1)$ is $\underline{\text{min}}$ $\xLeftarrow{\text{soc}}$ $h$ $\underline{\text{convex}}$ $\Leftarrow$ $H(h)$ pos. semidefin. by RRC

$f_{min} = \underline{\underline{2}}$

b) $\min f(x,y,z) = x^2+y^2+z^2$ when $3x^2+2y^2+2z^2 \geq 12$

$\Updownarrow$

$\max -f = -x^2-y^2-z^2 \qquad -3x^2-2y^2-2z^2 \leq -12$

$\mathcal{L} = -x^2-y^2-z^2 - \lambda(-3x^2-2y^2-2z^2)$

$\mathcal{L}'_x = -2x + 3\lambda\cdot 2x = 0$

$\mathcal{L}'_y = -2y + 2\lambda \cdot 2y = 0$

$\mathcal{L}'_z = -2z + 2\lambda\cdot 2z = 0$

$3x^2+2y^2+2z^2 \geq 12$

$\lambda \geq 0,\ \lambda\cdot(3x^2+2y^2+2z^2-12) = 0$

$(x^*,y^*,z^*) = (2,0,0)$: $-4+4\cdot 3\lambda = 0$

$\lambda = +4/12 = +\underline{1/3}$

$\Downarrow$

$\underline{(2,0,0;1/3)}$ ord. cond. pt.

$h(x,y,z) = -\frac{1}{3}y^2 - \frac{1}{3}z^2$

$H(h) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -2/3 & 0 \\ 0 & 0 & -2/3 \end{pmatrix}$

$\left\{ \begin{array}{l} \lambda_1 = 0 \leq 0 \\ \lambda_2 = -2/3 \leq 0 \\ \lambda_3 = -2/3 \leq 0 \end{array} \right.$ neg. semidefin.

$\Downarrow$

$(2,0,0)$ is max for $-f$ $\xLeftarrow{\text{soc}}$ $h$ $\underline{\text{concave}}$

$\underline{(2,0,0)}$ is min for $f$

$f_{min} = \underline{4}$

**2.**

a) $xyz=1$: $\quad \text{rk}\begin{pmatrix} yz & xz & xy \end{pmatrix} = 1$

if $yz=0$, then $y=0$ or $z=0$ $\Rightarrow$ not adm.

That is: this set of eqn. has no solutions:

$$\left.\begin{array}{l} xyz=1 \\ yz=0 \\ xz=0 \\ xy=0 \end{array}\right\} \text{rk } J<1$$

NDCQ satisfied at all adm. pts.

b) $3x^2+3y^2+8z^2 \geq 1$:

$>1$ : no cond.

$=1$ : $\text{rk}\begin{pmatrix} 6x & 6y & 24z \end{pmatrix} = 1$

NDCQ fails if $x=y=z=0$, and this is not adm.

c) $x^3+y^3+z^3=0$: $\quad \text{rk}\begin{pmatrix} 3x^2 & 3y^2 & 3z^2 \end{pmatrix} = 1$

$x=y=z=0 \Rightarrow (0,0,0)$ is adm. and NDCQ fails

d) $xy-zw=1$
$x+y+z+w=4$

$$\text{rk}\begin{pmatrix} y & x & -w & -z \\ 1 & 1 & 1 & 1 \end{pmatrix} = 2$$

NDCQ fails: all 2-minors in $J$ are zero

$\begin{vmatrix} y & x \\ 1 & 1 \end{vmatrix} = 0 \qquad y-x=0 \qquad y=x$

$\begin{vmatrix} x & -w \\ 1 & 1 \end{vmatrix} = 0 \qquad x+w=0 \qquad x=-w$

$\begin{vmatrix} -w & -z \\ 1 & 1 \end{vmatrix} = 0 \qquad -w+z=0 \qquad z=w$

$\Downarrow$

$x=-w,\ y=-w$
$z=w,\ w=w$

NDCQ satisfied at all adm. pts. $\Leftarrow$ Not adm since $xy-zw=(-w)^2-w^2$
$=0 \neq 1$

3. a) $\max f(x,y,z) = 2 - x^2 + 4xy - 5y^2 + 2yz - 2z^2$ when $3x - 2y + z = 10$

$h = f - \lambda \cdot (3x - 2y + z)$

$h'_x = -2x + 4y - 3\lambda = 0$

$h'_y = 4x - 10y + 2z + 2\lambda = 0$

$h'_z = 2y - 4z - \lambda = 0$

$3x - 2y + z = 10$

$$\left(\begin{array}{cccc|c} -2 & 4 & 0 & -3 & 0 \\ 4 & -10 & 2 & 2 & 0 \\ 0 & 2 & -4 & -1 & 0 \\ 3 & -2 & 1 & 0 & 10 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -3 & 10 \\ 0 & -18 & -2 & 14 & -40 \\ 0 & 2 & -4 & -1 & 0 \\ 0 & -8 & -2 & 9 & -20 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -3 & 10 \\ 0 & 2 & -4 & -1 & 0 \\ 0 & 0 & -38 & 5 & -40 \\ 0 & 0 & -18 & 5 & -20 \end{array}\right)$$

$$\left(\begin{array}{cccc|c} 1 & 2 & 1 & -3 & 10 \\ 0 & 2 & -4 & -1 & 0 \\ 0 & 0 & -20 & 0 & -20 \\ 0 & 0 & -18 & 5 & -20 \end{array}\right)$$

$x = -2\cdot 9/5 - 1 + 3\cdot 2/5 + 10 = \underline{21/5}$

$y = (4\cdot 1 - 2/5)/2 = \underline{9/5}$

$z = \underline{1}$

$\lambda = \underline{-2/5}$

Res. cond. pt: $(21/5, 9/5, 1; -2/5)$

SOC: $h = h(x,y,z; -2/5)$

$$H(h) = H(f) = \begin{pmatrix} -2 & 4 & 0 \\ 4 & -10 & 2 \\ 0 & 2 & -4 \end{pmatrix}$$

$D_1 = -2$

$D_2 = 4$

$D_3 = -4\cdot 4 - 2\cdot(-4) = -8$

$h$ neg. defn $\underset{\text{SOC}}{\Rightarrow}$ $h$ concave $\Rightarrow (21/5, 9/5, 1)$ is max

$f_{max} = f(21/5, 9/5, 1) = 0$

b) $\max f(x,y,z,w) = xz + yw$ when $\begin{cases} x^2+y^2 \le 1 \\ 4z^2+9w^2 \le 36 \end{cases}$

$$L = xz + yw - \lambda_1(x^2+y^2) - \lambda_2(4z^2+9w^2)$$

**FOC:**

$L'_x = z - \lambda_1 \cdot 2x = 0$ $\qquad z = 2\lambda_1 x$

$L'_y = w - \lambda_1 \cdot 2y = 0$ $\qquad \Downarrow$ $\qquad w = 2\lambda_1 y$

$L'_z = x - \lambda_2 \cdot 8z = 0$ $\qquad x = 8\lambda_2 \cdot (2\lambda_1 x) = 16\lambda_1\lambda_2 x$ $\qquad \Downarrow$

$L'_w = y - \lambda_2 \cdot 18w = 0$ $\qquad y = 18\lambda_2(2\lambda_1 y) = 36\lambda_1\lambda_2 y$

**FOC:** $x(1-16\lambda_1\lambda_2) = 0$ $\qquad x = 0$ or $\lambda_1\lambda_2 = 1/16$

$y(1-36\lambda_1\lambda_2) = 0$ $\qquad y = 0$ or $\lambda_1\lambda_2 = 1/36$

a) $x=y=0$: $\quad z = 2\lambda_1 x = 0$, $\quad w = 2\lambda_2 y = 0$ $\Rightarrow$

$$\boxed{\begin{array}{l} (x,y,z,w) = (0,0,0,0) \\ (\lambda_1,\lambda_2) = (0,0) \quad \text{since } C \text{ are } < . \\ f = 0 \end{array}}$$

b) $x=0$, $\lambda_1\lambda_2 = 1/36$: $\quad z = 2\lambda_1 x = 0$, $\quad \lambda_1, \lambda_2 > 0 \Rightarrow C$ are $=$ $\begin{cases} x^2+y^2=1 : & y = \pm 1 \\ 4z^2+9w^2=36 : & w = \pm 2 \end{cases}$

$w = 2\lambda_1 y$: $\quad w=2,\ y=1,\ \lambda_1 = 1 \Rightarrow \lambda_2 = 1/36$

$$\Rightarrow \boxed{\begin{array}{c} (x,y,z,w;\lambda_1,\lambda_2) = (0,\pm1,0,\pm2;\ 1,\ 1/36) \\ \text{with } y=w \\ f = 2 \end{array}}$$

c) $y=0$, $\lambda_1\lambda_2 = 1/16$: $\quad w = y = 0$, $\quad \lambda_1,\lambda_2 > 0 \Rightarrow C$ are $=$ $\begin{cases} x^2+y^2=1 : & x = \pm 1 \\ 4z^2+9w^2=36 : & z = \pm 3 \end{cases}$

$z = 2\lambda_1 x \Rightarrow z = x$, $\lambda_1 = 3/2$ $\lambda_2 = 1/24$

$$\boxed{\begin{array}{c} (x,y,z,w;\lambda_1,\lambda_2) = (\pm1,0,\pm3,0;\ 3/2,\ 1/24) \\ \text{with } x=y \ . \quad f = 3 \end{array}}$$

best candidate for max

d) $\left.\begin{array}{l} \lambda_1\lambda_2 = 1/16 \\ \lambda_1\lambda_2 = 1/36 \end{array}\right\}$ impossible

$h = xz + yw - \frac{3}{2}(x^2+y^2) - \frac{1}{24}(4z^2+9w^2)$

$$H(h) = \begin{pmatrix} -3 & 0 & 1 & 0 \\ 0 & -3 & 0 & 1 \\ 1 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -3/4 \end{pmatrix}$$

since $-\frac{8}{24} = -\frac{1}{3}$, $-\frac{18}{24} = -\frac{3}{4}$

$D_1 = -3$

$D_2 = 9$

$D_3 = 0$

$D_4 = -3/4 \cdot 0 + 1 \cdot \begin{vmatrix} -3 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1/3 & 0 \end{vmatrix}$

$= 0 + 0 = 0$

$$\begin{vmatrix} -3 & 0 & 1 \\ 0 & -3 & 0 \\ 1 & 0 & -1/3 \end{vmatrix} = -3 \cdot 0 = 0$$

$\Delta_1 = -3, -3, -1/3, -3/4 \leq 0$

$\Delta_2 = 9, 0, 9/4, 1, 5/4, 1/4 \geq 0$

$\Delta_3 = 0, -15/4, 0, -5/12 \leq 0$

$\Delta_4 = 0 \geq 0$

$$\begin{vmatrix} -3 & 0 & 0 \\ 0 & -3 & 1 \\ 0 & 1 & -3/4 \end{vmatrix} = -3\left(5/4\right) = -\frac{15}{4}$$

$$\begin{vmatrix} -3 & 1 & 0 \\ 1 & -1/3 & 0 \\ 0 & 0 & -3/4 \end{vmatrix} = -\frac{3}{4}(0) = 0$$

$$\begin{vmatrix} -3 & 0 & 1 \\ 0 & -1/3 & 0 \\ 1 & 0 & -3/4 \end{vmatrix} = -\frac{1}{3}\left(5/4\right) = -\frac{5}{12}$$

By the SOC, it follows that $(\pm 1, 0, \pm 3, 0)$ is max since $h$ is concave

and

$f_{max} = 3$

$(-1, 0, -3, 0)$, $(1, 0, 3, 0)$ are max pts

$\cancel{h = xz + yw - \frac{3}{2}(x^2+y^2) - \frac{1}{24}(4z^2+9w^2)}$

$\cancel{H(h) = \begin{pmatrix} -3 & 0 & 2 & 0 \\ 0 & -3 & 0 & 2 \\ 2 & 0 & -1/3 & 0 \\ 0 & 2 & 0 & -18/24 \end{pmatrix}}$ $\cancel{D_1 = -3}$, $\cancel{D_2 = 9}$, $\cancel{D_3 = -3 \cdot (1-4) = 9}$

~~$h$ not concave $\Downarrow$ cannot use SOC~~

**Alternative method:**

EVT: $x^2+y^2 \le 1$, $4z^2+9w^2 \le 36$ $\Rightarrow$ $-1 \le x, y \le 1$, $-3 \le z \le 3$, $-2 \le w \le 2$    $D$ is bounded

NDCQ: $\left.\begin{array}{l} c_1 = \\ c_2 = \end{array}\right\}$ $\operatorname{rk}\begin{pmatrix} 2x & 2y & 0 & 0 \\ 0 & 0 & 8z & 18w \end{pmatrix} = 2$

$\operatorname{rk} \cdot < 2$: $x=y=0$ or $z=w=0$ not adm $\Rightarrow$ NDCQ holds

$\left.\begin{array}{l} c_1 = \\ c_2 < \end{array}\right\}$ $\operatorname{rk}(2x \;\; 2y \;\; 0 \;\; 0) = 1$

$\operatorname{rk} \cdot < 1$: $x=y=0 \Rightarrow$ not adm. $\Rightarrow$ NDCQ holds

$\left.\begin{array}{l} c_1 < \\ c_2 = \end{array}\right\}$ $\operatorname{rk}(0 \;\; 0 \;\; 8z \;\; 18w) = 1$

$\operatorname{rk} \cdot < 1$: $z=w=0 \Rightarrow$ not adm. $\Rightarrow$ NDCQ holds

$\left.\begin{array}{l} c_1 < \\ c_2 < \end{array}\right\}$ no cond.

NDCQ holds for adm. pts in all 4 cases.

Concl: Best ord. cond. pt is max.

$\Rightarrow$ $(1, 0, 3, 0;\ 3/2, 1/24)$, $(-1, 0, -3, 0;\ 3/2, 1/24)$ are max pts

$f_{max} = 3$