# GRA 6035 Mathematics

## Problems for Lecture 9

## Key problems

**Problem 1.**
Let $0 < p < 1$ be a probability, with $q = 1 - p$, and let $a, b > 0$ be parameters such that $ap - bq > 0$. We consider the function $f(x) = p\ln(1 + ax) + q\ln(1 - bx)$ and the unconstrained optimization problem $\max f(x)$.
a) Show that the optimization problem has a solution for each value of the parameters $a, b, p$.
b) Compute the solution when $a = 2$, $b = 1$, and $p = 0.40$. What is the maximal value of $f$ in this case?
c) Use the envelope theorem to compute $df^*(p)/dp$, and use this to estimate the new maximum value of $f$ when $p = 0.43$.

**Problem 2.**
We consider the constrained optimization problem $\max f(x, y, z) = 4x^3 - 2y^3 + z^3$ when $x^3 + y^3 + z^3 \leq 8$.
a) Find the best candidate point in this problem.
b) Explain why this point is **not** a maximum point.

**Problem 3.**
We consider the constrained optimization problem $\max f(x, y, z) = 2x^2 - 4y^2 - 2z^2$ when $x^4 + y^4 + z^4 \leq 16$.
a) Use the EVT to show that this problem has a maximum point.
b) Show that the NDCQ is satisfied at all admissible points.
c) Find the maximum point and maximum value of $f$.
d) Use the envelope theorem to estimate the new maximum value of $f$ when we
i) change the constraint to $x^4 + y^4 + z^4 \leq 20$ ii) change the objective function to $f(x, y, z) = x^2 - 4y^2 - 2z^2$
iii) change the constraint to $x^4 + y^4 + z^4 \leq 20$ and the objective function to $f(x, y, z) = x^2 - 4y^2 - 2z^2$

## Problems from the Digital Workbook

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|---|---|
| Exercise problems | 9.1 - 9.5 (full solutions in the workbook) |
| Exam problems | 9.9, 9.10, 9.11ac (full solutions in the workbook) |

## Problems from Differential Equations

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|---|---|
| Problems in the appendix | A.1 - A.10 (full solutions on the web page) |
| Revision | Revise integrals from the appendix (or notes from FORK 1003) |

## Answers to key problems

**Problem 1.**
b) $x^* = 0.10$, $f^* \cong 0.0097$ c) $df^*(p)/dp = 0.2877$, $f^*(0.43) \cong 0.0183$

**Problem 2.**
a) $(x, y, z; \lambda) = (2, 0, 0; 4)$ with $f(2, 0, 0) = 32$
b) $f \to \infty$ when $x = z = 0$ and $y \to -\infty$ and $(0, y, 0)$ is admissible when $y \leq -2$ (for example, $f(0, -3, 0) = 81 > 32$)

**Problem 3.**
c) $(x, y, z; \lambda) = (\pm 2, 0, 0; 1/4)$ with $f(\pm 2, 0, 0) = 8$
d) *i)* $f_{\max} \cong 9$ *ii)* $f_{\max} \cong 4$ *iii)* $f_{\max} \cong 5$

# Solution: Key problems – Lecture 9

**1.** a) $$f'(x) = \frac{p\cdot a}{1+ax} + \frac{(-q)b}{1-bx} = \frac{pa(1-bx) - bq(1+ax)}{(1+ax)(1-bx)}$$

$$= \frac{ap - bq - ab(p+q)x}{(1+ax)(1-bx)} = \frac{ap-bq-abx}{(1+ax)(1-bx)} = 0$$

$$\Updownarrow$$

$$ap - bq = abx$$

$$x = \frac{ap-bq}{ab} > 0 \quad \text{since } ap - bq > 0$$

$$f''(x) = \frac{-pa^2}{(1+ax)^2} + \frac{-qb^2}{(1-bx)^2} < 0 \Rightarrow f \text{ concave}$$

$$x^* = \frac{ap-bq}{ab} \quad \text{max point}$$

b) $\left.\begin{array}{l} a=2, b=1 \\ p=0.40 \; q=0.60 \end{array}\right\}$ $$x^* = \frac{2\cdot 0.4 - 1\cdot 0.6}{2\cdot 1} = \frac{0.2}{2} = 0.10 \quad \text{max point}$$

$$f^* = f(0.10) = 0.40\ln(1+2\cdot 0.10) + 0.60\cdot \ln(1-0.10)$$
$$= 0.40\ln(1.2) + 0.60\cdot\ln 0.90$$
$$\simeq 0.0097 \quad \text{max value}$$

c) $$\frac{df^*(p)}{dp} = \frac{\partial f}{\partial p}(x = x^*) \longleftarrow \frac{\partial f}{\partial p} = \ln(1+ax) - \ln(1-bx) \quad \text{since } q = 1-p$$

$$= \ln\left(1 + a\cdot\frac{ap-bq}{ab}\right) - \ln\left(1 - b\cdot\frac{ap-bq}{ab}\right)$$

$$= \ln\left(1 + \frac{ap-bq}{b}\right) - \ln\left(1 - \frac{ap-bq}{a}\right)$$

$\left.\begin{array}{l} a=2 \\ b=1 \\ p=0.40 \\ x^*=0.10 \end{array}\right\}$ $$= \ln(1+2\cdot 0.10) - \ln(1 - 1\cdot 0.10) = \ln(1.2) - \ln(0.9) \simeq 0.2877$$

**Estimate for $p = 0.43$:**

$$f^*(0.43) \simeq f^*(0.40) + 0.03\cdot\frac{df}{dp} \simeq 0.0097 + 0.03\cdot 0.2877 = 0.0183$$

**2.** max $f(x,y,z) = 4x^3 - 2y^3 + z^3$ when $x^3 + y^3 + z^3 \leq 8$

a) $L = 4x^3 - 2y^3 + z^3 - \lambda(x^3 + y^3 + z^3)$

FOC:

$$\begin{aligned} L'_x &= 12x^2 - \lambda \cdot 3x^2 = 0 & 3x^2(4-\lambda) &= 0 & x &= 0 \text{ or } \lambda = 4 \\ L'_y &= -6y^2 - \lambda\, 3y^2 = 0 & 3y^2(-2-\lambda) &= 0 & y &= 0 \text{ or } \cancel{\lambda = -2} \\ L'_z &= 3z^2 - \lambda \cdot 3z^2 = 0 & 3z^2(1-\lambda) &= 0 & z &= 0 \text{ or } \lambda = 1 \end{aligned}$$

CSC: $\lambda \geq 0$

C: $x^3 + y^3 + z^3 = 8$ if $\lambda > 0$

Candidate pts:

i) $x = y = z = 0$, $x^3 + y^3 + z^3 = 0 < 8 \Rightarrow \lambda = 0 \Rightarrow \boxed{(0,0,0;0) \quad f = 0}$

ii) $x = y = 0$, $z \neq 0$ and $\lambda = 1$:
$x^3 + y^3 + z^3 = z^3 = 8 \Rightarrow z = 2$ $\quad \boxed{(0,0,2;1) \quad f = 8}$

iii) $y = z = 0$, $x \neq 0$ and $\lambda = 4$:
$x^3 + y^3 + z^3 = 8 \Rightarrow x = 2$ $\quad \boxed{(2,0,0;4) \quad f = 32}$

iv) $\lambda = 4$ and $\lambda = 1$: impossible

Best candidate point: $(x,y,z;\lambda) = (2,0,0;4)$ $\quad f = 32$

b) There is no max since $f \to \infty$ when $x = 0$, $y \to -\infty$ and $z = 0$. For example,

$$f(0,-3,0) = 81 > 32$$

Moreover $(0,y,0)$ is admissible when $y \leq 2$.

3. max $f(x,y,z) = 2x^2 - 4y^2 - 2z^2$ when $x^4+y^4+z^4 \leq 16$

a) $x^4+y^4+z^4 \leq 16$ defines a bounded set, since $-2 \leq x, y, z \leq 2$.

$\Downarrow$ EVT

there is a max.

b) NDCQ:
a) $x^4+y^4+z^4 < 16$ : no condition
b) $x^4+y^4+z^4 = 16$ :

$\text{rk } J = \text{rk}\,(4x^3\; 4y^3\; 4z^3) = 1$

NDCQ fails if $x=y=z=0$, not adm.

NDCQ holds for all adm. pts.

c) From a), b) it follows that there is a max at a regular candidate pt, with FOC + C + CSC satisfied.

$L = 2x^2 - 4y^2 - 2z^2 - \lambda(x^4+y^4+z^4)$

FOC:

$L'_x = 4x - \lambda \cdot 4x^3 = 0 \qquad 4x(1-\lambda x^2) = 0$

$L'_y = -8y - \lambda \cdot 4y^3 = 0 \qquad 4y(-2-\lambda y^2) = 0$

$L'_z = -4z - \lambda \cdot 4z^3 = 0 \qquad 4z(-1-\lambda z^2) = 0$

C + CSC:

a) $x^4+y^4+z^4 < 16$ : $\lambda = 0 \Rightarrow x=y=z=0$ $\quad \boxed{(0,0,0;0) \quad f=0}$

b) $x^4+y^4+z^4 = 16$: $\lambda \geq 0$

$x=0$ or $\lambda = 1/x^2$

$y=0$ or $\lambda = -2/y^2 < 0 \Rightarrow y=0$

$z=0$ or $\lambda = -1/z^2 < 0 \Rightarrow z=0$

$x=0 \Rightarrow (x,y,z) = (0,0,0)$ not adm. impossible.

$\lambda = 1/x^2 \Rightarrow x^2 = 1/\lambda \Rightarrow x = \pm\sqrt{1/\lambda}$

$x^4+y^4+z^4 = (1/\lambda)^2 + 0 + 0 = 16 \Rightarrow \lambda^2 = 1/16 \Rightarrow \lambda = 1/4$

$\Rightarrow \boxed{(\pm 2, 0, 0; 1/4) \quad f=8}$ Best cand. pt $\Rightarrow f_{max} = 8$

$(x^*, y^*, z^*) = (\pm 2, 0, 0)$

d) $\max f(x,y,z) = ax^2 - 4y^2 - 2z^2$ when $g(x,y,z) = x^4 + y^4 + z^4 - b = 0$

<u>Note:</u> 1) $a = 2,\ b = 16$ : $f^*(2,16) = 8$

$x^*(2,16) = \pm 2 \quad y^* = z^* = 0 \quad \lambda^* = 1/4$

2) the problem has a max which is regular cond. pt. as long as $a, b > 0$.

<u>Envelope thm:</u>

i) $a = 2,\ b = 20$: $f^*(2,20) = 8 + 4 \cdot \frac{df^*}{db} = 8 + 4 \cdot \overbrace{\frac{1}{4}}^{\lambda^*} = \underline{\underline{9}}$

ii) $a = 1,\ b = 16$: $f^*(1,16) = 8 + (-1) \cdot \underbrace{(\pm 2)^2}_{(x^*)^2} = \underline{\underline{4}}$

iii) $a = 1,\ b = 20$: $f^*(1,20) = 8 + (-1) \cdot (\pm 2)^2 + 4 \cdot 1/4 = \underline{\underline{5}}$

<u>We use that:</u>

$$L = ax^2 - 4y^2 - 2z^2 - \lambda(x^4 + y^4 + z^4 - b)$$

$$\frac{\partial L}{\partial a} = x^2 \Rightarrow \frac{df^*}{da} = x^*(a,b)^2$$

$$\frac{\partial L}{\partial b} = \lambda \Rightarrow \frac{df^*}{db} = \lambda^*(a,b)$$

(Exact values: $f^*(2,20) = 4\sqrt{5} \simeq 8.94$)

$f^*(1,16) = 4$

$f^*(1,20) = 2\sqrt{5} \simeq 4.47$