Plan

1. First order differential equations
2. Separable differential equations
3. Linear first order differential equations
4. Exact differential equations

No time for exact diff. eqn. today, will be covered next week.

Plenary Session: Monday 17-20

Lecture 7-9

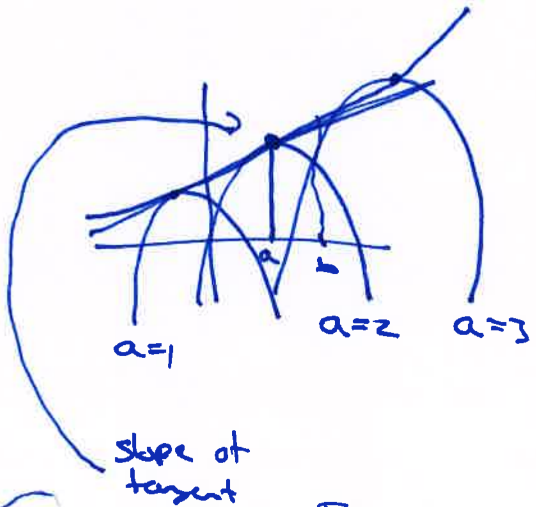
Review: Envelope Theorems

a) Unconstrained case:

$$\max/\min \; f(\underline{x}) = f(x_1, \dots, x_n)$$

$\underline{x}^*(a)$: max point

$$\underbrace{f^*(a) = f(\underline{x}^*(a))}_{\substack{\text{optimal value fn.} \\ = \text{envelope}}}$$



Env. Thm.

$$\boxed{\frac{df^*(a)}{da} = f'_a(\underline{x}^*(a))}$$

$$f^*(b) \simeq f^*(a) + \underbrace{(b-a)}_{\Delta a} \cdot \frac{df^*(a)}{da}$$

b) Constrained case: Lagrange / Kuhn-Tucker

$$\max/\min \; f(\underline{x}) \text{ with } \begin{cases} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases} \Big/ \begin{cases} g_1(x) \le a_1 \\ \vdots \\ g_m(x) \le a_m \end{cases}$$

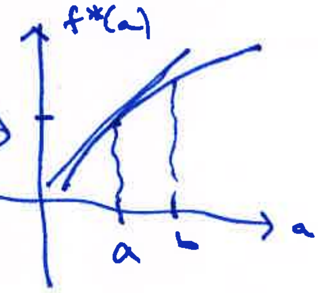
$g_1(x) - a_1 = 0$

$$L = f(\underline{x}) - \lambda_1(g_1(\underline{x}) - a_1) - \lambda_2(g_2(\underline{x}) - a_2) - \dots - \lambda_m(g_m(\underline{x}) - a_m)$$

$(\underline{x}^*(a); \lambda^*(a))$: optimal point such that FOC+C / FOC+C+CSC are satisfied

$\underbrace{f^*(a) = f(\underline{x}^*(a))}$

optimal value function = envelope



slope of tangent line

$$\boxed{\frac{df^*(a)}{da} = \mathcal{L}'_a(\underline{x}^*(a); \underline{\lambda}^*(a))}$$ Envelope Thm

$f^*(b) \simeq f^*(a) + \underbrace{(b-a)}_{\Delta a} \cdot \frac{df^*(a)}{da}$

c) Lagrange multipliers = special case

Ex: Key Problem 8.3b

$$\max\ xz + yw \quad \text{when} \quad \begin{cases} x^2+y^2 \le 1 = a \\ 4z^2+9w^2 \le 36 = b \end{cases}$$

$\underline{x}^*(1,36) = (\pm 1, 0, \pm 3, 0)$ with $x = z$

$\underline{\lambda}^*(1,36) = (3/2, 1/24)$

$f^*(1,36) = 3$

$$f^*(1.2, 36) \simeq \underset{3}{f^*(1,36)} + \underset{0.2}{\underbrace{(1.2-1)}_{\Delta a}} \cdot \underset{\mathcal{L}'_a(\underline{x}^*(a);\underline{\lambda}^*(a)) = \lambda_1^*(c) = 3/2}{\frac{df^*(a,b)}{da}} = \underline{\underline{3.3}}$$

$$f^*(1,30) \simeq \underset{3}{f^*(1,36)} + \underset{-6}{\underbrace{(30-36)}_{\Delta b}} \cdot \underset{\lambda_2^*(a,b) = \frac{1}{24}}{\frac{df^*(a,b)}{db}} = \underline{\underline{2.75}}$$

$\mathcal{L} = xz + yw - \lambda_1(x^2+y^2-a) - \lambda_2(4z^2+9w^2-b) = \ldots + \lambda_1 a + \lambda_2 b$

$\mathcal{L}'_a = \lambda_1 \qquad \mathcal{L}'_b = \lambda_2$

1 Differential equations

Defn: A differential equation is an equation such that:

i) a function $y = y(t)$ is the unknown

ii) it contains the derivative $y' = y'(t)$ or higher order derivatives $y'' = y''(t)$

Ex:

$$\left.\begin{array}{l} y' = 2y \\ y' = 2(y-t) \end{array}\right\} \text{first order} \qquad y'(t) = 2\,y(t)$$

$$y y'' - y' = t \;\Big\}\ \text{second order}$$

Defn: The <u>order</u> of a differential is the highest order of derivatives that appear in the eqn.

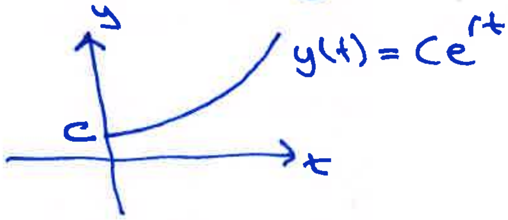
Why? First order differential eqn: $y' = \underbrace{F(t,y)}_{\text{modell for change in } y}$

Model: Simple exponential growth

$$y' = r \cdot y \qquad (r \text{ constant})$$

$$\Downarrow$$

$$y = C \cdot e^{rt}$$



$$y' = 2y, \quad y(0) = 250$$

Ex: $y' = f(t)$

$y = \int f(t)\,dt$

$y = F(t) + C$

$y' = 2t$

$y = \int 2t\,dt = t^2 + C$

$y = t^2 + C$

general solution (all solutions)

Initial value problem:

Ex: $y' = 2t$, $y(0) = 2$

$y' = 2t$

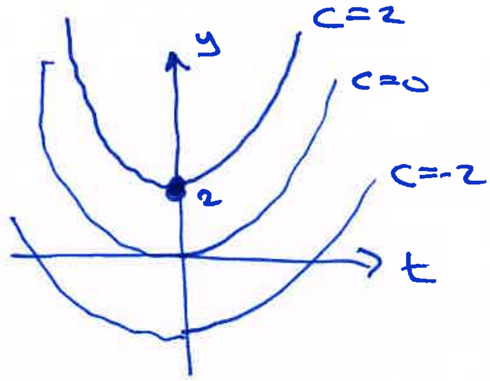
$y = \int 2t\,dt = t^2 + C$ general solution

$y(0) = 2$: $y = t^2 + C$

$2 = 0^2 + C$

$C = 2$

$y = t^2 + 2$ particular solution



particular solution = particular value of C

In general, a first order differential equation has a general solution that depend on one undetermined constant.

One ~~param~~ initial condition $\Rightarrow$ particular solution

② Separable differential equations

Defn: A separable differential eqn. can be written

$$\boxed{y' = f(t) \cdot g(y)}$$

Ex: $y' = 2y = \underline{2} \cdot \underline{y}$ separable

$y' + 2y = yt$

$\Updownarrow$

$y' = yt - 2y = \underbrace{(t-2)}_{f(t)} \cdot \underbrace{y}_{g(y)}$ separable

$y' = y + t$ not separable

Solution method:

Ex: $y' = t/y = t \cdot \frac{1}{y}$ or $| : \frac{1}{y}$ or $\cdot y$

$y \cdot y' = t$

$\int y \cdot y' \, dt = \int t \, dt$ $\leftarrow$ $\boxed{\begin{array}{c}\text{Substitution} \\ dy = y' \cdot dt\end{array}}$

$\int y \, dy = \int t \, dt$

$\frac{1}{2}y^2 + C_1 = \frac{1}{2}t^2 + C_2$ $\leftarrow$ implicit solution

$\frac{1}{2}y^2 = \frac{1}{2}t^2 + C_2 - C_1 \Rightarrow y^2 = t^2 + 2(C_2 - C_1)$

$$y^2 = t^2 + 2(C_2 - C_1)$$

$$y^2 = t^2 + C \qquad C = 2(C_2 - C_1)$$

$$y = \pm\sqrt{t^2 + C}$$ general solution
explicit form

General solution method:

$$y' = f(t) \cdot g(y) \quad | : g(y)$$

$$\frac{1}{g(y)} y' = f(t)$$

$$\int \frac{1}{g(y)} y' \, dt = \int f(t) \, dt$$

$$\boxed{\int \frac{1}{g(y)} \, dy = \int f(t) \, dt} \leftarrow$$ general solution
implicit form

- Find both integrals $\Rightarrow$ implicit form
- Solve for y $\Rightarrow$ Explicit form

3 Linear differential equations (of order one)

<u>Defn:</u> Linear first order differential eqn. can be written

$$\boxed{y' + a(t)y = b(t)} \iff y' = \underbrace{b(t) - a(t)y}_{\text{linear in } y}$$

standard form

BREAK

a) <u>Method:</u> Integrating factor

<u>Ex:</u> $y' = y + t$ (linear first order differential eqn.)

$\boxed{y' - y = t}$ $\quad a(t) = -1$, $b(t) = t$

$y' - y = t \quad | \cdot u = u(t)$ integrating factor

$uy' - uy = t \cdot u$

$\quad \parallel$

$(uy)' = t \cdot u$

$(uy)' = u \cdot y' + u' \cdot y = u \cdot y' - u\,y$

$\boxed{u' = -u}$

$u(t) = C \cdot e^{-t}$

<u>separable diff. eqn:</u>

$\frac{1}{u} \cdot u' = -1$

$\int \frac{1}{u}\, du = \int -1\, dt$

$\ln|u| + C_1 = -t + C_2$

$\ln|u| = -t + C_2 - C_1$

$|u| = e^{-t + C_2 - C_1}$

$u = \pm e^{-t} \cdot e^{C_2 - C_1}$

$u = C \cdot e^{-t}$

$\int (uy)'\, dt = \int t \cdot u(t)\, dt$

$uy = \int t \cdot u(t)\, dt$

$y = \frac{1}{u(t)} \int t\, u(t)\, dt$

Ex: $y' - y = t \quad | \cdot u = \underline{Ce^{-t}} = e^{-t} \quad (C=1)$

$e^{-t}y' - e^{-t}y = te^{-t}$

$(e^{-t} \cdot y)' = te^{-t}$

$e^{-t}y = \int te^{-t}\,dt$

$y = e^{t} \cdot \int te^{-t}\,dt$

$= e^{t}(-te^{-t} - e^{-t} + C)$

$$\underline{\underline{y = -t - 1 + Ce^{t}}}$$

$$\boxed{\begin{array}{ll} u = -e^{-t} & v = t \\ u' = e^{-t} & v' = 1 \end{array}}$$

integration by parts

$\int \underset{v}{t}\,\underset{u'}{e^{-t}}\,dt = u\cdot v - \int uv'\,dt$

$= -e^{-t} \cdot t - \int 1 \cdot (-e^{-t})\,dt$

$= -te^{-t} + \int e^{-t}\,dt$

$= \underline{-te^{-t} - e^{-t} + C}$

In general: Integrating factor

$y' + a(t)y = b(t) \quad | \cdot u = e^{\int a(t)\,dt}$

$(uy)' = b(t) \cdot u(t)$

$u \cdot y = \int b(t)\,u(t)\,dt$

$$\boxed{y = \frac{1}{u(t)} \int b(t)\,u(t)\,dt}$$

$$\boxed{\begin{array}{l} \underline{a(t) = a \text{ is constant}} \\ u = e^{at} \end{array}}$$

Formula: Integrating factor $u = e^{\int a(t)\,dt}$

Ex: $y' + ay = b \quad | \cdot u = e^{at}$

$(y \cdot e^{at})' = be^{at}$

$y \cdot e^{at} = \int be^{at} = \frac{b}{a}e^{at} + C$

$$\underline{\underline{y = \frac{b}{a} + Ce^{-at}}}$$

b) Method: Super position principle

Ex: $y' - y = t$

Superposition principle: $y = y_h + y_p$

y_h: general solution of the homogeneous eqn. $y' - y = 0$

y_p: particular solution of $y' - y = t$

y_h: $y' - y = 0$

$y' = y \Rightarrow y = Ce^t$

a(t) = a constant

easy to find $y_h = Ce^{-at}$

y_p: $y' - y = t$

$y = At$: $A - At = t$

$y = At + B$: $A - (At+B) = t$

$\underbrace{A - B}_{\text{const}} \underbrace{- At}_{t\text{-term}} = t$

$y_p = -t - 1$

$A = -1$ $B = -1$

$\Downarrow$

$y = y_h + y_p = Ce^t - t - 1$

Constant coefficients: $a = a(t)$ is constant

$\Rightarrow$ superposition often easier to use than integrating factor

$y' + ay = b(t)$

y_h: $y' + ay = 0$

$\boxed{y_h = Ce^{-at}}$

Method: Separable $y' = -ay \Rightarrow y = Ce^{-at}$