Note: Change of dates!

Plenary Session 4

11/11 at 17-20 in A1-040.

Plan

1 Exact differential equations
2 Equilibrium states and stability
3 Second order differential equations

Review:

First order differential equations: $y' = F(t,y)$

General solution: contains one undetermined coeff.

i) Separable: $y' = f(t) \cdot g(y)$

$\frac{1}{g(y)} y' = f(t)$

$\boxed{\int \frac{1}{g(y)} dy = \int f(t) dt}$ implicit form

Solve for $y \to$ explicit form

ii) Linear first order: $\boxed{y' + a(t) y = b(t)} \Leftrightarrow y' = b(t) - a(t) y$

Integrating factor:

$u = e^{\int a(t) dt}$ $(u' = u \cdot a(t))$

$u y' + u a(t) y = b(t) u(t)$

$(u \cdot y)' = b(t) u(t)$

$u y = \int b(t) u(t) dt$

$\boxed{y = \frac{1}{u} \int u(t) b(t) dt}$

Superposition principle

$y = y_h + y_p$

y_h: homogeneous sol. = gen. solution of $y' + a(t) y = 0$

y_p: particular solution of $y' + a(t) y = b(t)$

① Exact differential equations

Defn: A first order differential equation

$$\boxed{p(t,y) + q(t,y)\cdot y' = 0}$$

is exact if there is a function $h(t,y)$ such that i) $h'_t = p$ ii) $h'_y = q$

Solution method: $\boxed{h(t,y) = C}$

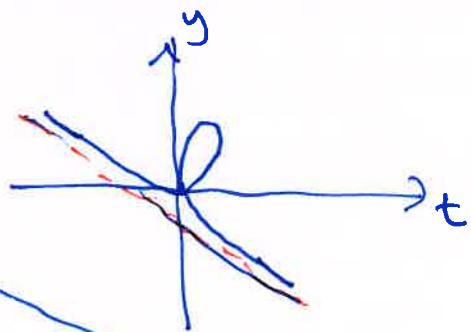
Exact:

$h'_t + h'_y \cdot y' = 0$

$\frac{d}{dt}\left(h(t,y)\right) = 0$ total derivative → $h(t,y) = C$ general solution

Ex: Let us try to find a differential equation with solution

$$t^3 - 3ty + y^3 = 0$$



$$(t^3 - 3ty + y^3)'_t = (0)'_t$$

$$3t^2 - 3(1\cdot y + ty') + 3y^2\cdot y' = 0$$

$$3t^2 - 3(1\cdot y + t\cdot y') + 3y^2\cdot y' = 0$$

$$\underbrace{3t^2 - 3y}_{h'_t} \underbrace{-3ty' + 3y^2y'}_{\underbrace{(-3t+3y^2)y'}_{h'_y}} = 0$$

$h(t,y) = C$ level curve

$$\boxed{h'_t + h'_y\cdot y' = 0}$$

$$\Updownarrow$$

$$y' = -\frac{h'_t}{h'_y}$$

$$\left(y' = -\frac{f'_x}{f'_y}\right)$$

Ex: $\boxed{(3t^2-3y)+(3y^2-3t)y'=0} \iff y' = \dfrac{3y-3t^2}{3y^2-3t}$

$p(t,y) = 3t^2-3y$, $q(t,y) = 3y^2-3t$

not linear
not separable

Exact? ① $h'_t = 3t^2-3y$
② $h'_y = 3y^2-3t$

$\boxed{\begin{array}{c} h'_t + h'_y \cdot y' = 0 \\ h(t,y) = C \end{array}}$

① $h'_t = 3t^2-3y$

$h = \underline{t^3 - 3yt + C(y)}$

② $h'_y = 3y^2-3t$

$(t^3-3yt+C(y))'_y = 3y^2-3t$

$0 \underline{-3t} + C'(y) = 3y^2 \underline{-3t}$

$C'(y) = 3y^2$

$\underline{C(y) = y^3}$

$\underline{h = t^3-3yt+y^3}$

Conclusion: i) the differential is exact
ii) the general solution

$h(t,y) = C$

$\underline{\underline{t^3-3yt+y^3 = C}}$

general solution in implicit form.

<u>Result:</u>

$$\boxed{\begin{gathered} p(t,y) + q(t,y)\,y' = 0 \quad \text{is exact} \\ \Updownarrow \\ p'_y = q'_t \end{gathered}}$$

<u>Explanation:</u> If it is exact: $p = h'_t$
$q = h'_y$

$\Downarrow$

$$p'_y = (h'_t)'_y = h''_{ty}$$
$$q'_t = (h'_y)'_t = h''_{yt}$$

BREAK

② <u>Equilibrium states and stability</u>

First order diff. eqn.: $y' = F(t,y)$

<u>Defn:</u> It is called <u>autonomous</u> if $y' = F(y)$

<u>Ex:</u> $y' = 2y$ $\qquad$ $y' = 5y(1 - y/10)$

<u>Defn.</u> An <u>equilibrium state</u> is a constant solution of $y' = F(y)$; that is: $\underline{F(y) = 0}$

<u>Ex:</u>

$F(y) = 0$

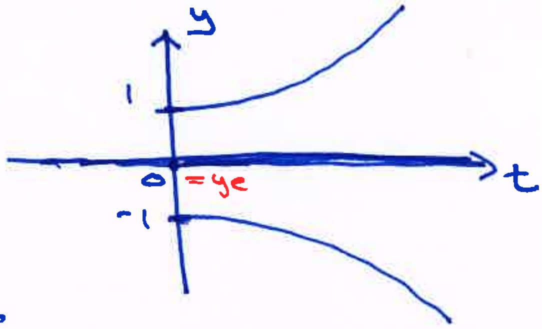
a) $y' = 2y$: $2y = 0 \Rightarrow \underline{\underline{y_e = 0}}$

b) $y' = 5y(1 - y/10)$: $5y(1 - y/10) = 0$

$\underline{\underline{y_e = 0}}$, $\underline{\underline{y_e = 10}}$

<u>Ex:</u> $y' = 2y$

$y = \underline{C \cdot e^{2t}}$



<u>t=0:</u> $y = C \cdot e^0 = C$

$y_0 = C$

$y_e = 0$ is unstable

<u>$y_0 = 1$:</u> $y = e^{2t}$

$y_0 = -1$: $y = -e^{2t}$

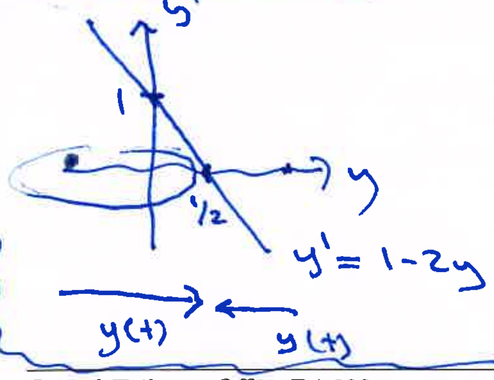
<u>$y_0 = y_e = 0$:</u> $y = 0 \cdot e^{2t} = 0$

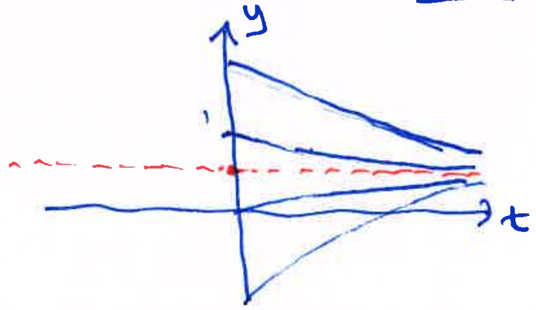
<u>Stability:</u> An equilibrium state is <u>unstable</u> if $y(t)$ moves away from y_e when $y_0 \neq y_e$.
It is <u>stable</u> if $y(t)$ moves towards y_e when $y_0 \neq y_e$ is close to y_e.
It is <u>globally asymptotically stable</u> if $y(t)$ moves toward y_e for all $y_0 \neq y_e$.

<u>Ex:</u> $y' = 1 - 2y$

Eq. states: $1 - 2y = 0$ $y_e = \underline{1/2}$

<u>Phase diagram:</u>





$y_e = 1/2$ globally asymp. stable

Theorem:

If $y = y_e$ is an equilibrium state of $y' = F(y)$, then we have:

$$F'(y_e) > 0 \Rightarrow y_e \text{ is } \underline{\text{unstable}}$$

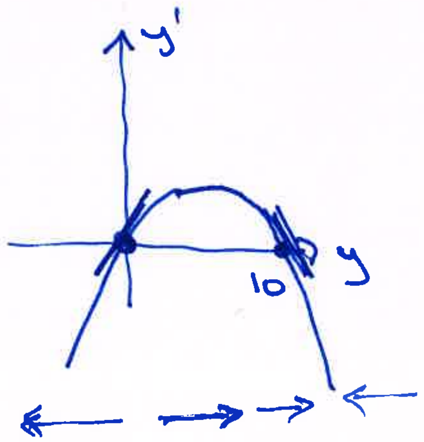
$$F'(y_e) < 0 \Rightarrow y_e \text{ is } \underline{\text{stable}}$$

$5y - \frac{5}{10}y^2$

$y' = 5y(1 - y/10)$

$\underline{y_e}$: $\underline{y=0}$, $\underline{y=10}$

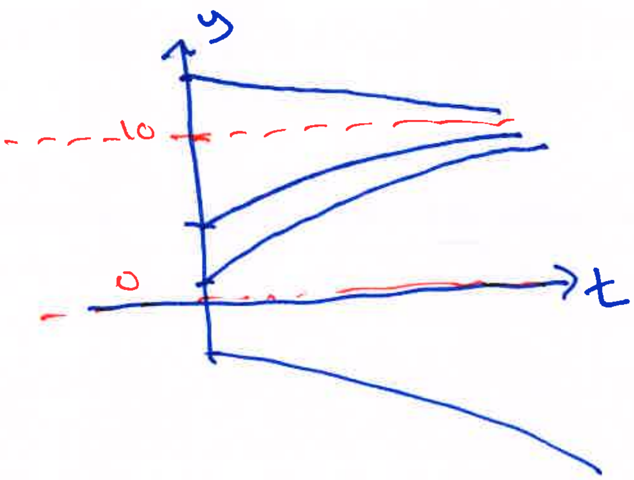
Ex:



$y_e = 0$ unstable

$y_e = 10$ stable

(but not globally asymp. stable)



gl. as. Stable

unstable

stable, but not gl. as. stable

BREAK

3 Second order differential equations

Ex:

$$y'' = 12$$

$$y' = 12t + C \quad \leftarrow \quad \int 12\,dt$$

$$y = \int 12t + C\,dt$$

$$y = \underline{\underline{6t^2 + Ct + D}} \qquad \text{general solution}$$

two undetermined coeff.

Ex: $y'' = 12$, $y(0) = 0$, $y'(0) = 1$

i) $y'(0) = 12 \cdot 0 + C = 1$
$\underline{\underline{C = 1}}$

ii) $y(0) = 6 \cdot 0^2 + C \cdot 0 + D = 0$
$\underline{\underline{D = 0}}$

Particular solution:

$$y = \underline{\underline{6t^2 + t}}$$

Defn: A second order differential is linear with constant coefficients if it has the form

$$\boxed{y'' + ay' + by = f(t)}$$

where a, b are constants, f(t) is any expression in t.

Method: Superposition principle

$y = y_h + y_p$, where

y_h: general solution of $y'' + ay' + by = 0$ (homogeneous eqn.)

y_p: particular solution of $y'' + ay' + by = f(t)$.

Ex: $y'' - 5y' + 6y = 2e^t$

y_h: $y'' - 5y' + 6y = 0$

Method: Characteristic equation:

$y = e^{rt}$ $y' = e^{rt} \cdot r$ $y'' = e^{rt} \cdot r^2$

$(r^2 \cdot e^{rt}) - 5(r e^{rt}) + 6(e^{rt}) = 0$

$e^{rt} \cdot (r^2 - 5r + 6) = 0$

$\boxed{\begin{array}{c} y'' + ay' + by = 0 \\ \Updownarrow \\ r^2 + ar + b = 0 \end{array}}$

$r^2 - 5r + 6 = 0$ Characteristic equation.

$r = \dfrac{5 \pm \sqrt{25 - 4 \cdot 6}}{2 \cdot 1} = \dfrac{5 \pm 1}{2}$

$r = 3, 2$

$y_h = C_1 \cdot e^{3t} + C_2 e^{2t}$

General case: (for y_h)

$y'' + ay' + by = 0$

$r^2 + ar + b = 0$

$r = \dfrac{-a \pm \sqrt{a^2 - 4b}}{2}$

i) $\Delta = a^2 - 4b > 0$: two roots $r_1 \neq r_2 \Rightarrow y_h = C_1 \cdot e^{r_1 t} + C_2 \cdot e^{r_2 t}$

ii) $\Delta = a^2 - 4b = 0$: $r_1 = r_2 = -a/2 \Rightarrow y_h = C_1 e^{r_1 t} + C_2 t e^{r_1 t}$

iii) $\Delta = a^2 - 4b < 0$: no real roots

$$r = \frac{-a \pm \sqrt{a^2-4b}}{2} = \underbrace{\left(-\frac{a}{2}\right)}_{\alpha} \pm \underbrace{\frac{\sqrt{4b-a^2}}{2}}_{\beta} \cdot \sqrt{-1} \Rightarrow y_h = e^{\alpha t} \cdot (C_1 \cdot \cos(\beta t) + C_2 \sin(\beta t))$$

$$y = y_h + y_p = C_1 e^{3t} + C_2 e^{2t} + e^t$$

y_p:

Ex: $y'' - 5y' + 6y = 2e^t$

$y = y_h + y_p$, $y_h = C_1 e^{3t} + C_2 e^{2t}$

Method: Undetermined coefficients

i) "Guess" a solution $y(t)$

- Same "form" as $f(t)$
- Include undetermined coeff.

ii) compute y', y'' and substitute it in the diff. eqn.

Ex: $f(t) = 2e^t$ $\quad y = A \cdot e^t$? $\quad y = 2e^{rt}$?

$y = A \cdot e^{rt}$?

compute: $f'(t), f''(t)$ $\quad f' = 2e^t \quad f'' = 2e^t$

$\Rightarrow y = Ae^t$ $\quad y' = Ae^t \quad y'' = Ae^t$

guess

$y'' - 5y' + 6y = 2e^t$

$(Ae^t) - 5(Ae^t) + 6(Ae^t) = 2e^t$

$(A - 5A + 6A)e^t = 2e^t$ $\rightarrow$ $2Ae^t = 2e^t$

$A = 1$

$y_p = e^t$

Ex: $y'' + 6y' + 9y = t$

Superposition: $y = y_h + y_p$

y_h: $y'' + 6y' + 9y = 0$

$r^2 + 6r + 9 = 0$

$r = \frac{-6 \pm \sqrt{6^2 - 4 \cdot 9}}{2} = \frac{-6 \pm 0}{2} = -3$ (mult. two)

$y_h = C_1 \cdot e^{-3t} + C_2 t e^{-3t} = \underline{(C_1 + C_2 t) e^{-3t}}$

y_p: $y'' + 6y' + 9y = t$ "Guess": $f(t) = t \leftarrow$ t-term

$f' = 1 \leftarrow$ const. term

$f'' = 0$

$0 + 6A + 9(At + B) = t \longleftarrow$ $y = \underline{At + B}$

$(9A)t + (6A + 9B) = t$ $y' = A$

$y'' = 0$

$9A = 1$, $6A + 9B = 0$

$\underline{A = 1/9}$ $9B = -6/9$ $y_p = \underline{\frac{1}{9}t - \frac{6}{81}}$

$\underline{B = -6/81}$

Solution: $y = y_h + y_p = \underline{(C_1 + C_2 t)e^{-3t} + \frac{1}{9}t - \frac{6}{81}}$