Plan

1. Systems of linear differential equations
2. Linear difference equations

Monday: Plenary Session 4 (1700–2000)

Lecture 13: Problems. Review.
Lecture 14: Final exam 11/2018.

Review:

a) **Exact differential equations**

$$h'_t + h'_y \cdot y' = 0 \implies h(t,y) = C$$

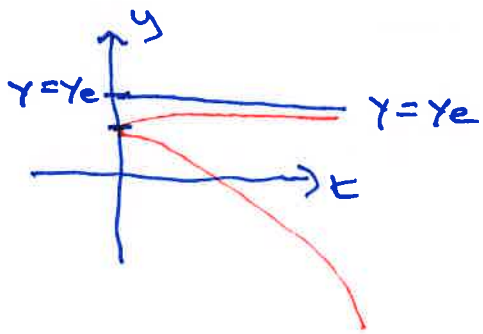
for a function $h = h(t,y)$.

$$\boxed{p + q \cdot y' = 0 \text{ exact} \iff p'_y = q'_t}$$

b) **Equilibrium states and stability:**

$y' = F(y)$ autonomous

Eq. state: $y = y_e$ s.t. $F(y_e) = 0$ (constant solutions)

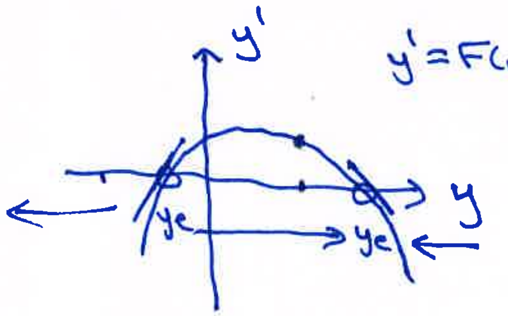


Stability:

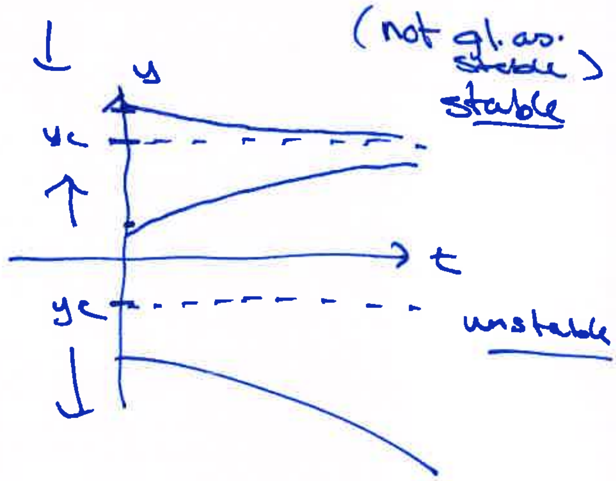
$y = y_e$ is stable if $y(t)$ approaches y_e if y_0 is close to y_e.
$y = y_e$ is unstable otherwise.

Thm: $F'(y_e) < 0 \implies y_e$ stable
$F'(y_e) > 0 \implies y_e$ unstable

$y = y_e$ is <u>globally asymptotically stable</u> if $y(t)$ approaches y_e for any starting point y_0.



phase diagram



c) <u>Second order linear diff. eq.</u>

$y'' + ay' + by = f(t)$ (a, b constants)

$y = y_h + y_p$ (superposition)

<u>y_h</u>: $y'' + ay' + by = 0$

$r^2 + ar + b = 0$ (char. eqn.)

- two sol. $r_1 \neq r_2$: $\underline{y_h = C_1 \cdot e^{r_1 t} + C_2 \cdot e^{r_2 t}}$
- one double root $r_1 = r_2$: $y_h = C_1 \cdot e^{r_1 t} + C_2 \cdot t e^{r_1 t}$

<u>y_p</u>: method of undetermined coeff.

"Guess" $y(t)$, adjust so that it fits.
- same form as $f(t)$
- undetermined coeff.

If it doesn't work, try to <u>multiply with t</u>.

① Systems of linear differential equations

Ex:
$$y_1' = 2y_1 + 5y_2$$
$$y_2' = 5y_1 + 2y_2$$
coupled

$$y_1' = 3y_1$$
$$y_2' = \quad\quad 2y_2$$
decoupled

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$\boxed{\mathbf{y}' = A\cdot \mathbf{y}}, \quad \mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$y_1' = 3y_1$$
$$\Updownarrow$$
$$y_1' - 3y_1 = 0$$
$$y_1 = \underline{C_1 e^{3t}}$$

$$y_2' = 2y_2$$
$$y_2' - 2y_2 = 0$$
$$y_2 = \underline{C_2 \cdot e^{2t}}$$

$$y_1 = C_1 e^{3t}$$
$$y_2 = C_2 e^{2t}$$

$$A = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$$
$$\mathbf{y}' = A\cdot\mathbf{y}$$

Diagonalize A: $A = \begin{pmatrix} 2 & 5 \\ 5 & 2 \end{pmatrix}$

$$\begin{vmatrix} 2-\lambda & 5 \\ 5 & 2-\lambda \end{vmatrix} = 0$$

$$\lambda^2 - 4\lambda - 21 = 0$$

$$\lambda = \frac{4 \pm \sqrt{16 - 4(-21)}}{2} = \frac{4 \pm 10}{2}$$

$$\underline{\lambda_1 = 7} \qquad \lambda_2 = \underline{-3}$$

E_7: $\begin{pmatrix} -5 & 5 \\ 5 & -5 \end{pmatrix}$ $\quad \underline{v_1} = y\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

E_{-3}: $\begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}$ $\quad \underline{v_2} = y\cdot\begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$D = \begin{pmatrix} 7 & 0 \\ 0 & -3 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$$

$$\boxed{P^{-1}AP = D}$$

Transform system into
$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ 0 & -3 \end{pmatrix}\cdot\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$
by choosing $\underline{u}$ from the matrix P.

$$\underline{y}' = A\cdot\underline{y} \qquad D = P^{-1}AP \quad \Big\} \quad \underline{u}' = D\cdot\underline{u}$$

$$y_1' = 2y_1 + 5y_2$$
$$y_2' = 5y_1 + 2y_2$$

$$u_1' = 7u_1$$
$$u_2' = \quad -3u_2$$

Put: $P.\underline{u} = \underline{y}$
$$\Updownarrow$$
$$\underline{u} = P^{-1}.\underline{y}$$

$$P^{-1} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$\boxed{\begin{aligned} u_1 &= \tfrac{1}{2}(y_1+y_2) \\ u_2 &= \tfrac{1}{2}(-y_1+y_2) \end{aligned}}$$

Check that $\underline{u}' = D\cdot\underline{u}$:

$$\underline{u}' = (P^{-1}.\underline{y})' = P^{-1}.\underline{y}' = P^{-1}.A\underline{y}$$
$$D.\underline{u} = (P^{-1}A\cancel{P}).(\cancel{P^{-1}}\underline{y}) = P^{-1}A\underline{y} \quad) \text{ ok}$$

$$\begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 7 & 0 \\ 0 & -3 \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \qquad \begin{aligned} u_1' &= 7u_1 \\ u_2' &= -3u_2 \end{aligned} \qquad \begin{aligned} u_1 &= C_1 e^{7t} \\ u_2 &= C_2\cdot e^{-3t} \end{aligned}$$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = P.\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} C_1 e^{7t} \\ C_2 e^{-3t} \end{pmatrix}$$
$$= C_1 e^{7t}.\begin{pmatrix} 1 \\ 1 \end{pmatrix} + C_2 e^{-3t}.\begin{pmatrix} -1 \\ 1 \end{pmatrix} = \begin{pmatrix} C_1 e^{7t} - C_2 e^{-3t} \\ C_1 e^{7t} + C_2 e^{-3t} \end{pmatrix}$$

$$\underline{\underline{\underline{y} = C_1\cdot\begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{7t} + C_2\begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-3t}}}$$

You can write this:

$$y_1 = C_1 e^{7t} - C_2 e^{-3t}$$
$$y_2 = C_1 e^{7t} + C_2 e^{-3t}$$

Solutions of linear systems of differential eqns:

If $\underline{y}' = A \cdot \underline{y}$, where A is an $n \times n$ diagonalizable matrix and $\underline{y} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$, $\underline{y}' = \begin{pmatrix} y_1' \\ \vdots \\ y_n' \end{pmatrix}$, then

$$\boxed{\underline{y} = C_1 \cdot \underline{v_1} e^{\lambda_1 t} + C_2 \cdot \underline{v_2} e^{\lambda_2 t} + \dots + C_n \cdot \underline{v_n} e^{\lambda_n t}}$$

with: $\lambda_1, \lambda_2, \dots, \lambda_n$ = eigenvalues of A
$\underline{v_1}, \underline{v_2}, \dots, \underline{v_n}$ = linearly independent eigenvectors of A

Link with second order linear diff.eqns:

$$y'' - 5y' + 6y = 0$$

$$\left.\begin{aligned} u_1 &= y \\ u_2 &= y' \end{aligned}\right\} \qquad \begin{aligned} y'' &= 5y' - 6y \\ u_2' &= 5u_2 - 6u_1 \end{aligned}$$

$$\begin{aligned} u_1' &= \phantom{-6u_1 +{}} u_2 \\ u_2' &= -6u_1 + 5u_2 \end{aligned} \qquad \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -6 & 5 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

Eigenvalues:

$$\begin{vmatrix} 0-\lambda & 1 \\ -6 & 5-\lambda \end{vmatrix} = 0 \qquad \begin{aligned} -\lambda(5-\lambda) + 6 &= 0 \\ \lambda^2 - 5\lambda + 6 &= 0 \end{aligned}$$

$$\lambda = 2, \quad \lambda = 3$$

$\underline{\lambda = 2:}$ $\begin{pmatrix} -2 & 1 \\ \cancel{-6} & \cancel{3} \end{pmatrix}$ $-2x + y = 0$ $\quad$ x free, $y = 2x$ $\quad$ $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 2x \end{pmatrix} = \underline{x \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$

$\underline{\lambda = 3:}$ $\begin{pmatrix} -3 & 1 \\ \cancel{-6} & \cancel{2} \end{pmatrix}$ $-3x + y = 0$ $\quad$ x free, $y = 3x$ $\quad$ $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x \\ 3x \end{pmatrix}$, $\underline{x \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}}$

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \underline{\underline{C_1 \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t} + C_2 \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix} e^{3t}}} = \begin{pmatrix} y \\ y' \end{pmatrix}$$

$$y = \underline{\underline{C_1 e^{2t} + C_2 e^{3t}}}$$

$$y' = \underline{2C_1 e^{2t} + 3C_2 e^{3t}}$$

This method can be used to solve

$$y^{(n)} + a_1 y^{(n-1)} + \dots + a_{n-1} y' + a_n y = 0$$

using $n \times n$ linear systems of diff. eqn.

② Difference equations

Ex: $y_{t+1} = 1{,}05\, y_t$

Solution:

Sequence

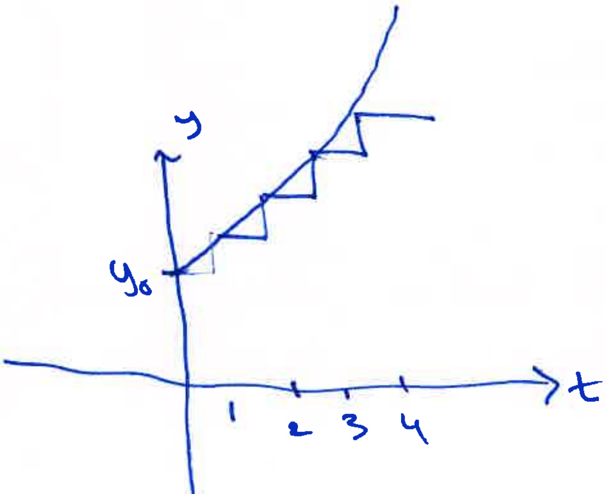
$\{y_t\} = \{y_0, y_1, y_2, \ldots\}$

(discrete time)

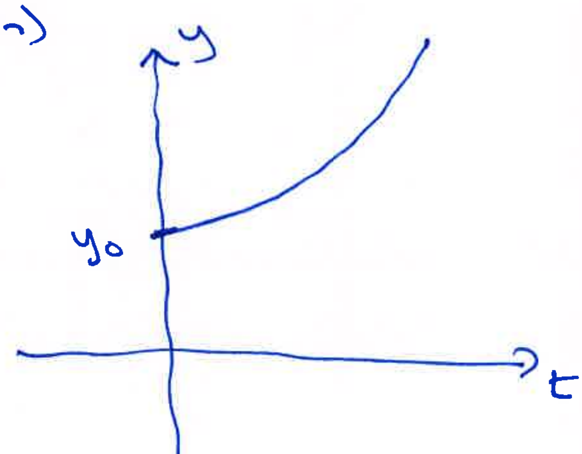
$$y_1 = 1{,}05\, y_0$$
$$y_2 = 1{,}05\, y_1 = 1.05^2 y_0$$
$$\vdots$$
$$y_{t+1} = 1{,}05\, y_t = 1{,}05^{t+1} \cdot y_0$$

Solution: $\underline{\underline{y_t = C \cdot 1{,}05^t}}$ $\quad ((C = y_0))$

(closed form solution)



(discrete time)



(continuous time)

$$y_{t+1} = 1.05\, y_t$$
$$y_{t+1} - y_t = 0.05\, y_t$$
$$y_t = C \cdot 1{,}05^t$$

$$y' = 0.05\, y$$
$$y = C \cdot e^{0.05t}$$

i) Linear first order difference equation:

$y_{t+1} = a \cdot y_t$ (a constant) (homogeneous case)

Solution: $y_t = \underline{\underline{C \cdot a^t}}$

$y_{t+1} - a \cdot y_t = f_t$ (general case)

Ex: $y_{t+1} - 2y_t = t$

$y_t = y_t^h + y_t^p$
$= \underline{\underline{C \cdot 2^t + (-t-1)}}$

y_t^h: $y_{t+1} - 2y_t = 0$ $\longrightarrow$ $r - 2 = 0$ characteristic equ.

$y_{t+1} = 2y_t$ $\underline{r = 2}$

$y_t^h = \underline{C \cdot 2^t}$ $y_t^h = \underline{\underline{C \cdot 2^t}}.$

r^t, $\underline{\text{not}}$ e^{rt} for difference eqn.

y_t^p: $y_{t+1} - 2y_t = t$ $f_t = t$, $f_{t+1} = t+1$

$$\begin{cases} y_t = \underline{At+B} \leftarrow \text{guess} \\ y_{t+1} = A \cdot (t+1) + B \\ \quad = At + (A+B) \end{cases}$$

$(At + A + B) - 2(At + B) = t$

$\underline{-A}t + \underline{(A-B)} = t$

$-A = 1$ $\quad A - B = 0$

$A = -1$ $\quad B = -1$

$y_t^p = \underline{\underline{-t-1}}$

ii) Second order linear difference equations

$$y_{t+2} + a \cdot y_{t+1} + b y_t = f_t \quad \begin{cases} a, b \text{ constants} \\ f_t : \text{expression in } t \end{cases}$$

Ex: $y_{t+2} - y_t = 1$ $\qquad y_t = y_t^h + y_t^p$

y_t^h: $\quad y_{t+2} - y_t = 0 \qquad y_t = r^t \qquad r^{t+2} - r^t = 0$

$r^t \underline{(r^2 - 1)} = 0$

$r^2 - 1 = 0$

$r = \pm 1 \implies y_t^h = C_1 \cdot 1^t + C_2 \cdot (-1)^t = \underline{C_1 + C_2 \cdot (-1)^t}$

y_t^p: $\quad y_{t+2} - y_t = 1 \qquad y_t = A$

$y_{t+2} = A$

$A - A = 1$

$0 \cdot A = 1$ $\qquad$ not possible; multiply with t

$y_t = \underline{A \cdot t}$

$(At + 2A) - At = 1 \qquad y_{t+2} = A(t+2) = \underline{At + 2A}$

$\underbrace{0 \cdot At}_{0} + \underbrace{2A}_{1} = 1$

$A = \underline{1/2} \qquad y_t^p = \frac{1}{2} t$

$$y_t = y_t^h + y_t^p = \underline{C_1 + C_2 \cdot (-1)^t + \tfrac{1}{2} t}$$

Characteristic equation:

$$y_{t+2} + a y_{t+1} + b y_t = 0$$

$$r^2 + ar + b = 0 \implies r = \frac{-a \pm \sqrt{a^2 - 4b}}{2}$$

$r_1 \neq r_2$: two roots — $y_t = C_1 \cdot r_1^t + C_2 \cdot r_2^t$

$r_1 = r_2$: one double root — $y_t = C_1 \cdot r_1^t + C_2 \cdot t \cdot r_1^t$

no real roots: there is a solution, using complex numbers

Ex: $y_{t+2} = y_{t+1} + y_t$, $y_0 = y_1 = 1$

$y_2 = 2$
$y_3 = 3$
$y_4 = 5$
$y_5 = 8$
$y_6 = 13$
$\vdots$

Fibonacci sequence

$$y_{t+2} - y_{t+1} - y_t = 0$$

$$r^2 - r - 1 = 0$$

$$r = \frac{1 \pm \sqrt{1 - 4(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}$$

$$y_t = C_1 \cdot \left(\frac{1+\sqrt{5}}{2}\right)^t + C_2 \cdot \left(\frac{1-\sqrt{5}}{2}\right)^t$$

iii) Systems of linear difference equations

$$\underline{y}_{t+1} = A \cdot \underline{y}_t : \quad \begin{pmatrix} y_{1,t+1} \\ y_{2,t+1} \\ \vdots \\ y_{n,t+1} \end{pmatrix} = A \cdot \begin{pmatrix} y_{1,t} \\ y_{2,t} \\ \vdots \\ y_{n,t} \end{pmatrix}$$

Solution:

$$\underline{y}_t = C_1 \cdot \underline{v}_1 \cdot \lambda_1^t + C_2 \cdot \underline{v}_2 \lambda_2^t + \dots + C_n \cdot \underline{v}_n \cdot \lambda_n^t$$

Assumption: A is diagonalizable

$\lambda_1, \dots, \lambda_n$: eigenvalues of A

$\underline{v}_1, \dots, \underline{v}_n$: lin. independent eigenvectors of A