## Plan

1 Revision: Vectors and matrices (Lecture 1-5)
2 Revision: Optimization problems (Lecture 6-9)
3 Revision: Differential equations (Lecture 10-12)

Monday: TA session (Oct 18)

Office hours: - away Monday Oct 18
- At the office all other days up to the final exam

Exam Subs: Final exam 11/2018 (Lecture 14)
Final exam 12/2016 (TA session)

## Review:

### i) Systems of differential equations

$$\begin{pmatrix} y_1' \\ y_2' \\ \vdots \\ y_n' \end{pmatrix} = A \cdot \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix} \quad \text{or} \quad \underline{y}' = A \cdot \underline{y}$$

$A$: $n \times n$ matrix

**Assume:** $A$ is diagonalizable with eigenvalues $\lambda_1, \lambda_2 \ldots, \lambda_n$ and eigenvectors $\underline{v}_1, \ldots, \underline{v}_n$
(the eigenvectors are lin. independent with $A \cdot \underline{v}_i = \lambda_i \cdot \underline{v}_i$)

**General solution:**

$$\underline{y} = c_1 \cdot \underline{v}_1 \cdot e^{\lambda_1 t} + c_2 \cdot \underline{v}_2 e^{\lambda_2 t} + \ldots + c_n \cdot \underline{v}_n \cdot e^{\lambda_n t}$$

### ii) Difference equations:

a) Linear first order: $y_{t+1} + a \cdot y_t = f_t$

b) Second order linear: $y_{t+2} + a y_{t+1} + b y_t = f_t$

**Solution method:** $y_t = y_t^h + y_t^p$

$y_t^h$: Characteristic equation ($y_t = r^t$)

$y_t^p$: method of undetermined coefficients (guess $y_t$ and check)

a) $r + a = 0$: $r = -a$, $y_t^h = c_1 \cdot (-a)^t$

b) $r^2 + ar + b = 0$

$r_1 \neq r_2$: $y_t^h = c_1 \cdot r_1^t + c_2 \cdot r_2^t$

$r_1 = r_2$: $y_t^h = c_1 \cdot r_1^t + c_2 \cdot t \cdot r_1^t$

## c) Systems of difference equations:

$$\underline{Y}_{t+1} = A \cdot \underline{y}_t \quad \text{, or} \quad \begin{pmatrix} Y_{1,t+1} \\ Y_{2,t+1} \\ \vdots \\ Y_{n,t+1} \end{pmatrix} = A \cdot \begin{pmatrix} Y_{1,t} \\ Y_{2,t} \\ \vdots \\ Y_{n,t} \end{pmatrix}$$

With same assumptions as in the case of systems of differential equations

(A diagonalizable; $\lambda_1, \ldots, \lambda_n$ eigenvalues; $\underline{v}_1, \ldots, \underline{v}_n$ eigenvectors)

$$\underline{Y}_t = C_1 \cdot \underline{v}_1 \lambda_1^t + C_2 \cdot \underline{v}_2 \lambda_2^t + \ldots + C_n \cdot \underline{v}_n \lambda_n^t$$

Markov chain: $\lambda_1 = 1$, $|\lambda_2| \ldots |\lambda_n| < 1$

$$\underline{Y}_t = C_1 \cdot \underline{v}_1 \cdot 1^t + \left( C_2 \underline{v}_2 \lambda_2^t + \ldots + C_n \underline{v}_n \lambda_n^t \right)$$

$\downarrow$

$C_1 \underline{v}_1$ as $t \to \infty$

$\uparrow$

the unique eigenvector with $\lambda = 1$ which is a stable vector

# ① Vectors and matrices

- linear systems, Gaussian elimination
- rank, determinants
- diagonalization, eigenvalues and eigenvectors
- linear independence, base, dimension, span
- nullspace Null(A), column space Col(A) = span($\underline{v}_1, .., \underline{v}_n$) (span of col. vectors)
  (Null(A): all solutions of $A \cdot \underline{x} = \underline{0}$)
- definiteness, principal minors

**Rank:** — rank = # pivot position = maximal order of a non-zero minor

= maximal # of linearly independent column vectors

**Linear independence and span:**

* $\{\underline{v}_1, .., \underline{v}_n\}$ are lin. independent if no vector is a linear combination of the others
  $\Leftrightarrow$
  $x_1\underline{v}_1 + .. + x_n\underline{v}_n = 0$ has only the trivial solution $\underline{x} = \underline{0}$.
* base for V: a set of lin. independent vector that span V

V = Null(A): $\underline{x} = t_1 \cdot \underline{v}_1 + .. + t_k \cdot \underline{v}_k \Rightarrow \{\underline{v}_1, .., \underline{v}_k\}$ is a base
($t_1, .., t_k$ are the free variables)

$V = Col(A)$ : Find the pivot positions in $A = (\underline{v}_1 \dots \underline{v}_n)$

$= span(\underline{v}_1, \dots \underline{v}_n)$ ↓ the corresponding vectors form a basis

- dimension : $\dim V$ = # vectors in a basis of $V$

$V = Null(A)$: $\dim V = n - rk(A)$ ← # free variables

$V = col(A)$: $\dim Col(A) = rk\, A$

- definiteness:

A pos. def. $\iff D_1, D_2 \dots, D_n > 0$

A neg. def. $\iff D_1 < 0, D_2 > 0, D_3 < 0, \dots$

$(D_1 < 0, D_2 > 0, \dots)$

REC: If A has rank $r < n$ and $D_1, D_2 \dots, D_r > 0$ then A is positive semidefin. (neg semidefin.)

- diagonalizable :

A symmetric $\Rightarrow$ A diagonalizable

A has { i) n eigenvalues (counted with mult.); ii) n lin. independent eigenvectors } $\iff$ A diagonalizable

Ex: $A = \begin{pmatrix} 4 & 7 & -1 \\ -2 & 0 & 3 \\ 2 & 7 & 2 \end{pmatrix}$

a) Eigenvalues:

$$\begin{vmatrix} 4-\lambda & 7 & -1 \\ -2 & -\lambda & 3 \\ 2 & 7 & 2-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)\cdot[-\lambda(2-\lambda)-21] - 7\cdot[-2(2-\lambda)-6] + (-1)\cdot[-14-2(-\lambda)] = 0$$

$$(4-\lambda)(\lambda^2-2\lambda-21) - 7(2\lambda-10) - (2\lambda-14) = 0$$

$$(4-\lambda)(\lambda^2-2\lambda-21) - 16\lambda + 84 = 0$$

$$4\lambda^2 - 8\lambda - 84 - \lambda^3 + 2\lambda^2 + 21\lambda - 16\lambda + 84 = 0$$

$$-\lambda^3 + 6\lambda^2 - 3\lambda = 0$$

$$-\lambda(\lambda^2 - 6\lambda + 3) = 0$$

$\sqrt{4}\cdot\sqrt{6}$

$\lambda = 0$ or $\lambda^2 - 6\lambda + 3 = 0$

$$\lambda = \frac{6 \pm \sqrt{36 - 4\cdot 3}}{2} = \frac{6 \pm \sqrt{24}}{2}$$

$\lambda_1 = 0$ $\lambda_2 = 3 + \sqrt{6}$ $\lambda_3 = 3 - \sqrt{6}$

If constant term $\neq 0$: Integer solution must divide the constant term.

Ex: $x^3 - 3x^2 + 3x + 1 = 0$ → try $x = \pm 1$

$x = 1$: ok

$x = -1$: $-1 - 3 - 3 - 1 \neq 0$

$x^3 - 3x^2 + 3x - 1 = 0$

$(x-1)\cdot(x^2 + 2x + 1) = 0$

polyn. division: $(x^3 - 3x^2 + 3x - 1) : (x-1)$

b/ $E_0$: $\lambda_1 = 0$ $\quad E_0 = \text{Null}(A-\lambda I) = \text{Null}(A)$ when $\lambda = 0$

$$\left(\begin{array}{ccc|c} 4 & 7 & -1 & 0 \\ -2 & 0 & 3 & 0 \\ 2 & 7 & 2 & 0 \end{array}\right) \to \left(\begin{array}{ccc} -2 & 0 & 3 \\ 4 & 7 & -1 \\ 2 & 7 & 2 \end{array}\right) \begin{matrix} \\ \cdot 2 \\ \cdot 1 \end{matrix} + 1$$

$$\to \left(\begin{array}{ccc} -2 & 0 & 3 \\ 0 & 7 & 5 \\ 0 & 7 & 5 \end{array}\right) \cdot -1 \to \left(\begin{array}{ccc} -2 & 0 & 3 \\ 0 & 7 & 5 \\ 0 & 0 & 0 \end{array}\right)$$

$-2x + 3z = 0 \quad \Rightarrow x = 3z/2$

$7y + 5z = 0 \quad \Rightarrow y = -5z/7$

$z = z$

z free

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3z/2 \\ -5z/7 \\ z \end{pmatrix} = z \cdot \begin{pmatrix} 3/2 \\ -5/7 \\ 1 \end{pmatrix} \qquad \text{Base: } \left\{ \begin{pmatrix} 3/2 \\ -5/7 \\ 1 \end{pmatrix} \right\}$$

or: $= z/14 \cdot \begin{pmatrix} 21 \\ -15 \\ 14 \end{pmatrix}$ Base: $\left\{ \begin{pmatrix} 21 \\ -15 \\ 14 \end{pmatrix} \right\}$

## (2) Optimization problems

- Stationary points, partial derivatives
- convex/concave fu., Hessian matrix
- Second derivative test
- closed and bounded sets, extreme value theorem
- Lagrange problems: FOC+C = Lagrange cond.
  Kuhn-Tucker pts.: FOC+C+CSC = KT cond.
- techniques for solving systems of equations and inequalities
- Second order condition (SOC), NDCQ
- envelope theorems

- Convex/concave: $f$ convex $\Longleftrightarrow$ $H(f)$ is pos. semidefin. at all pts. $\underline{x}$.

  $f$ concave $\Longleftrightarrow$ $H(f)$ is neg. semidefin. at all pts. $\underline{x}$.

  $f$ convex (concave) $\Rightarrow$ any stationary pt is global min. (global max)

- Second derivative test:

  $\underline{x}^*$ stat. pt. $\Longrightarrow$

  | | | | |
  |---|---|---|---|
  | $H(f)(x^*)$ | pos. defn. | $\Rightarrow$ $\underline{x}^*$ | local min |
  | " | neg. defn | " | local max |
  | " | indefinite | " | saddle pt. |

- SOC: If $(\underline{x}^*; \underline{\lambda}^*)$ is a candidate pt that satisfies FOC+C / FOC+C+CSC, then:

  $h(\underline{x}) = L(\underline{x}; \underline{\lambda}^*)$ concave $\Rightarrow \underline{x}^*$ is max

  — " — convex $\Rightarrow \underline{x}^*$ is min

- Methods for Lagrange/KT problems

  (1) Find ordinary candidate pts FOC+C / FOC+C+CSC

  (2) Conclude using: — SOC (check one pt.)
  — EVT (check all cand. pts + NDCQ)

- Envelope thm:

  unconstrained case: $\frac{df^*(a)}{da} = f'_a(\underline{x}^*(a))$

  constrained case: $\frac{df^*(a)}{da} = L'_a(\underline{x}^*(a); \underline{\lambda}^*(a))$

# Problem 3, Key Problems Lecture 9

max $f(x,y,z) = 2x^2 - 4y^2 - 2z^2$ s.t. $x^4+y^4+z^4 \le 16$

bounded ✓

$x^4 \le 16$

$-2 \le x \le 2$
$-2 \le y \le 2$
$-2 \le z \le 2$

$L = 2x^2 - 4y^2 - 2z^2 - \lambda(x^4+y^4+z^4-16)$

FOC:
$L'_x = 4x - \lambda \cdot 4x^3 = 0$
$L'_y = -8y - \lambda\, 4y^3 = 0$
$L'_z = -4z - \lambda \cdot 4z^3 = 0$

KT conditions:

a) Binding case: $x^4+y^4+z^4 = 16$

CSC: $\lambda \ge 0$

FOC: $4x - \lambda \cdot 4x^3 = 0$
$-8y - \lambda\, 4y^3 = 0$
$-4z - \lambda \cdot 4z^3 = 0$

$1 - \lambda x^2 = 0$

$4x(1 - \lambda \cdot x^2) = 0$

$x=0$ or $\lambda = 1/x^2$

$4y(-2 - \lambda y^2) = 0$

$y=0$ or ~~$\lambda = -2/y^2$~~

$4z(-1 - \lambda z^2) = 0$

$z=0$ or ~~$\lambda = -1/z^2$~~

i) $x=y=z=0$: $\lambda \ge 0$, $0^4+0^4+0^4 = 0 \ne 16$

no cand pts.

ii) $\lambda = 1/x^2$, $y=z=0$: $\lambda \ge 0$, $x^4+0^4+0^4 = 16$

$x = \pm\sqrt[4]{16} = \pm 2$

$(2,0,0;\ 1/4)$, $f=8$ $\quad$ $(-2,0,0;\ 1/4)$ $f=8$

b) Non-binding: $x^4+y^4+z^4 < 16$

CSC: $\lambda = 0$

FOC: $4x = 0$
$-8y = 0$
$-4z = 0$

$(x,y,z) = (0,0,0)$ $\lambda = 0$

⇓

Cand pts: $(0,0,0;\ 0)$ $f=0$

---

NDCQ fails: no pts

⇓ EVT

$f_{max} = 8$ at $(\pm 2,0,0)$ $\lambda = 1/4$

Max $f = 2x^2 - 4y^2 - 2z^2$ s.t. $x^4 + y^4 + z^4 = 16$ ← a, $= 20$ ← a

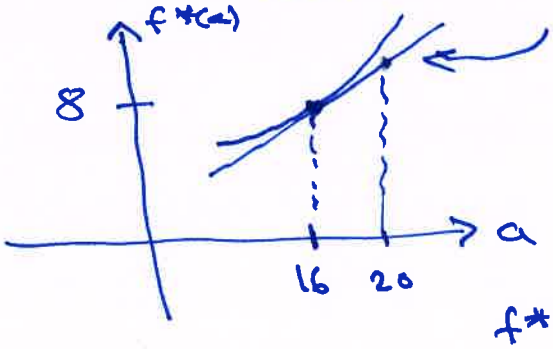
b: $= 1 \cdot x^2$

Original KT problem: $f^* = 8$ at $(x^*, y^*, z^*) = (\pm 2, 0, 0)$

$\lambda^* = 1/4$

a: $16 \to 20$ : $L = 2x^2 - 4y^2 - 2z^2 - \lambda(x^4 + y^4 + z^4 - a)$

$L'_a = \lambda \Rightarrow$ Env. Thm. $\frac{df^*(a)}{da} = \lambda^*(a)$




slope of tangent line $= \lambda^*(16) = 1/4$

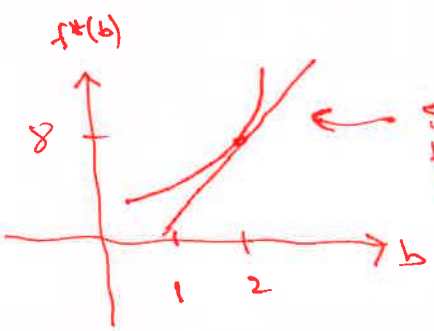
$\Delta a = 1$

$f^*(20) \simeq f^*(16) + 4 \cdot 1/4$

$8 \qquad = 9$

b: $2 \to 1$ : $L = bx^2 - 4y^2 - 2z^2 - \lambda(x^4 + y^4 + z^4 - 16)$

$L'_b = x^2 \Rightarrow$ Env. Thm. $\frac{df^*(b)}{d(b)} = x^*(b)^2$




slope of tangent line $= x^*(2)^2 = (\pm 2)^2 = 4$

$8 \qquad \Delta b = -1$

$f^*(1) \simeq f^*(2) + (-1) \cdot 4 = 4$

## (3) Differential equations (Lecture 10 - 12)

- first order linear, integrating factor, superposition
- separable $\int \frac{1}{g(y)}\,dy = \int f(t)\,dt$
- exact $h'_t + h'_y \cdot y' = 0 \Rightarrow h = C$
- second order linear, superposition
- systems of diff. equations
- difference equations, systems of diff. eqns
- equilibrium states, stability