Plan

1. Final exam 11/2018 Problem 1-5
2. Revision

Main topics:
- Matrices and vectors
- Optimization
- Differential / difference equations

Remember:
- course evaluation

Office hours:
- the rest of the day today (Friday)
- Monday / Tuesday all day (from around 10.00)

Exam	Final exam in GRA 6035 Mathematics
Date	November 27th, 2018 at 0900 - 1200

This exam consists of 12+1 problems (one additional problem is for extra credits, and can be skipped). Each problem has a maximal score of 6p, and 72p (12 solved problems) is marked as 100% score.

You must give reasons for your answers. Precision and clarity will be emphasized when evaluating your answers.

Question 1.

We consider the matrix A given by

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 8 \\ -1 & 0 & 1 & 0 \\ 0 & 8 & 0 & -2 \end{pmatrix}$$

(a) **(6p)** Compute the rank of A. How many free variables does $A \cdot \mathbf{x} = \mathbf{0}$ have?
(b) **(6p)** Find Null(A), and determine its dimension.
(c) **(6p)** Determine the definiteness of A.

Question 2.

(a) **(6p)** Find the general solution of the differential equation $y'' - 12y' + 20y = 3e^{-t}$.
(b) **(6p)** Find the general solution of the following system of differential equations:

$$\begin{aligned} y_1' &= 3y_1 + 4y_2 \\ y_2' &= 4y_1 - 3y_2 \end{aligned}$$

(c) **(6p)** Find the equilibrium states of the autonomous differential equation $y' = 0.15y(1-y/200)$ and determine their stability. Are any of the equilibrium states globally asymptotically stable?

Question 3.

We consider the function $f(x, y, z) = 16 - x^4 - 2x^2 - 3y^2 + 6xz - 6z^2 + 10z$.

(a) **(6p)** Find all stationary points of f with $x = 1$.
(b) **(6p)** Show that f has a global maximum point, and find the maximal value of f.
(c) **(6p)** Use the envelope theorem to estimate $\max\,(16 - x^4 - 2x^2 - 3y^2 + 6xz - 6z^2 + 11z)$.

Question 4.

We consider the Kuhn-Tucker problem

$$\max\ f(x, y, z) = 3x^2 - y^2 - 2z^2 \text{ subject to } 2x^4 + 2y^4 + z^4 \leq 18$$

(a) **(6p)** Write down the Kuhn-Tucker conditions for this problem.
(b) **(6p)** Find all points $(x, y, z; \lambda)$ that satisfy the Kuhn-Tucker conditions.
(c) **(6p)** Show that the best candidate points from (b) are the maximum points, and use this to determine the maximum value.

Question 5.

Extra credit (6p) Solve the logistic differential equation $y' = 0.15y(1 - y/200)$, and determine the time it takes for the system to reach 90% of the carrying capacity when $y_0 = 50$.

① Final exam 11/2018

1. $A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 8 \\ -1 & 0 & 1 & 0 \\ 0 & 8 & 0 & -2 \end{pmatrix}$

a) $\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 8 \\ -1 & 0 & 1 & 0 \\ 0 & 8 & 0 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & -2 \end{pmatrix} \begin{matrix} \\ \\ -4 \\ \end{matrix}$

$\rightarrow \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 8 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -34 \end{pmatrix}$

$\text{rk } A = 3$

\# free variables $= 4 - \text{rk}(A) = 4 - 3 = 1$
in $A \cdot \underline{x} = \underline{0}$.

b) Null(A): Solution of $A \cdot \underline{x} = \underline{0}$.

$\left(\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 2 & 0 & 8 & 0 \\ 0 & 0 & 0 & -34 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$

echelon form of augmented matrix

$x - z = 0 \Rightarrow x = z$

$2y + 8w = 0 \quad y = 0$

$-34w = 0 \Rightarrow w = 0$

z free

$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} z \\ 0 \\ z \\ 0 \end{pmatrix} = z \cdot \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$, z free

Base: $\left\{ \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \right\}$

$\dim \text{Null}(A) = 1$

c) Definiteness:

$$A = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 2 & 0 & 8 \\ -1 & 0 & 1 & 0 \\ 0 & 8 & 0 & -2 \end{pmatrix}$$

Leading principal minors:

$D_1 = 1$

$D_2 = 2$

$D_3 = 2 \cdot (1 \cdot 1 - (-1)\cdot(-1)) = 0$

$D_4 = 0$ since $\operatorname{rk} A = 3$

All principal minors:

$\Delta_1 = 1, 2, 1, -2$

both positive and negative Δ_1's

$\Downarrow$

A indefinite

RRC: $\operatorname{rk}(A) = 3$

$D_1 = 1$

$D_2 = 2$

$D_3 = 0$

Cannot use this

2. a) $y'' - 12y' + 20y = 3e^{-t}$

Second order linear diff. eqn. $\Rightarrow$ Superposition:

$y = y_h + y_p$

y_h: $y'' - 12y' + 20y = 0$

char. eqn: $r^2 - 12r + 20 = 0$

$$r = \frac{12 \pm \sqrt{144 - 80}}{2} = \frac{12 \pm 8}{2}$$

$r_1 = 10$, $r_2 = 2$ $\longrightarrow$ $y_h = C_1 \cdot e^{10t} + C_2 e^{2t}$

$\underline{y_p:}$ $\quad y'' - 12y' + 20y = \boxed{3e^{-t}}$

$f(t) = 3e^{-t}$
$f'(t) = -3e^{-t}$
$f''(t)\ 3e^{-t}$

$$\left\{ \begin{array}{l} \boxed{y = A \cdot e^{-t}} \\ y' = -Ae^{-t} \\ y'' = Ae^{-t} \end{array} \right.$$

$$(Ae^{-t}) - 12(-Ae^{-t}) + 20\cdot(Ae^{-t}) = 3e^{-t} \Leftarrow$$

$$(\underbrace{A + 12A + 20A}_{33A})e^{-t} = 3e^{-t}$$

$$33A = 3 \Rightarrow A = 3/33 = \underline{1/11} \Rightarrow y_p = \underline{\tfrac{1}{11}e^{-t}}$$

$\underline{\text{Concl:}}$ $\quad y = y_h + y_p = \underline{\underline{C_1 e^{10t} + C_2 e^{2t} + \tfrac{1}{11}e^{-t}}}$

b) $\quad y_1' = 3y_1 + 4y_2$
$\quad y_2' = 4y_1 - 3y_2$

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = A\cdot\begin{pmatrix} y_1 \\ y_2 \end{pmatrix},\ A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix}$$

Symmetric
$\Downarrow$
$\underline{\text{diagonalizable}}$

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = C_1 \cdot \underline{v_1} e^{\lambda_1 t} + C_2 \underline{v_2} e^{\lambda_2 t}$$
$$= \underline{\underline{C_1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{5t} + C_2 \cdot \begin{pmatrix} -1/2 \\ 1 \end{pmatrix} e^{-5t}}} \Leftarrow$$

$\underline{\text{Eigenvalues of } A:}$ $\quad \begin{vmatrix} 3-\lambda & 4 \\ 4 & -3-\lambda \end{vmatrix} = 0$

$$\lambda^2 + (-25) = 0$$
$$\lambda^2 = 25 \qquad \underline{\lambda = \pm 5}$$

$\underline{\lambda_1 = 5,\ \lambda_2 = -5}$

$\underline{\text{Eigenvectors of } A:}$

$E_5:$ $\quad \begin{pmatrix} -2 & 4 \\ 4 & -8 \end{pmatrix} \xrightarrow{\cdot 2} \begin{pmatrix} -2 & 4 \\ 0 & 0 \end{pmatrix}$

$-2x + 4y = 0 \qquad x = 2y,\ y = \text{free} \qquad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2y \\ y \end{pmatrix} = \underline{y\cdot\begin{pmatrix} 2 \\ 1 \end{pmatrix}} \qquad \underline{v_1} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

$\underline{E_{-5}:}$ $\quad \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix} \xrightarrow{-1/2} \begin{pmatrix} 8 & 4 \\ 0 & 0 \end{pmatrix}$

$8x + 4y = 0 \qquad x = -\tfrac{1}{2}y,\ y \text{ free}: \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}y \\ y \end{pmatrix} = y\cdot\begin{pmatrix} -1/2 \\ 1 \end{pmatrix} \qquad \underline{v_2} = \begin{pmatrix} -1/2 \\ 1 \end{pmatrix}$

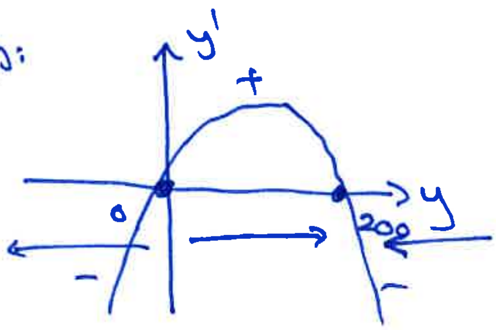
c) $y' = 0.15y(1 - y/200)$

Eq. states: $0.15y(1 - y/200) = 0$

$0.15y = 0$ or $1 - y/200 = 0$

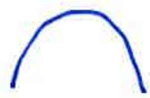
$\underline{y_e = 0}$ $\qquad$ $y_e = \underline{200}$

Stability:



$y' = 0.15y(1 - y/200)$

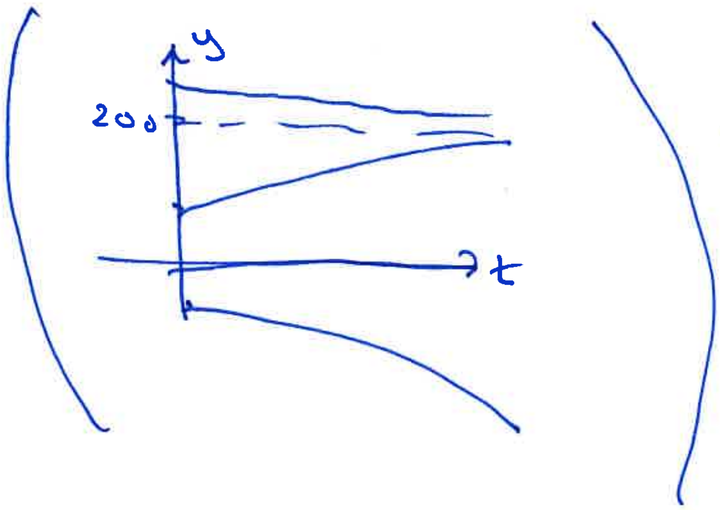
$= 0.15y - \frac{0.15}{200}y^2$



$y_e = 0$ is unstable

$y_e = 200$ is stable but not globally as. stable

(since $y_0 < 0$ means that $y(t)$ does not ~~approach~~ approach $y_e = 200$)



3. $f(x,y,z) = 16 - x^4 - 2x^2 - 3y^2 + 6xz - 6z^2 + 10z$

a) Stationary pts with $x=1$:

$$\text{FOC} \left\{ \begin{array}{l} f'_x = -4x^3 - 4x + 6z = 0 \\ f'_y = -6y = 0 \\ f'_z = 6x - 12z + 10 = 0 \end{array} \right.$$

$$\begin{aligned} f(1,0,4/3) &= 16-1-2-0 \\ &\quad + 6\cdot 4/3 - 6\cdot(4/3)^2 \\ &\quad + 10\cdot 4/3 \\ &= 21 - \frac{16\cdot \cancel{6}^{2}}{\cancel{9}3} + \frac{40}{3} \\ &= 21 + 8/3 = \underline{71/3} \end{aligned}$$

$\underline{x=1}$: $-8+6z=0$ $\quad z = 8/6 = \underline{4/3}$

$-6y=0 \implies y=\underline{0}$

$16-12z=0 \qquad 16-12\cdot(4/3) = 0$ ok.

Stat pts with $x=1$: $(x,y,z) = \underline{\underline{(1,0,4/3)}}$

b) Check if f is concave:

$$H(f) = \begin{pmatrix} -12x^2-4 & 0 & 6 \\ 0 & -6 & 0 \\ 6 & 0 & -12 \end{pmatrix}$$

$D_1 = -12x^2-4 < 0$ for all (x,y,z)

$D_2 = -6D_1 > 0$ — " —

$(= 72x^2+24)$

$D_3 = -6\cdot\left[(-12x^2-4)(-12) - 36\right]$

$= -6\,(144x^2+48-36)$

$= -6\cdot(144x^2+12) < 0$ for all (x,y,z)

$H(f)(x,y,z)$ negative defn. for all x,y,z

$\Downarrow$

f concave

$\Downarrow$

$\underline{\underline{(1,0,4/3) \text{ global max}}}$

$f_{max} = f(1,0,4/3) = \underline{71/3}$

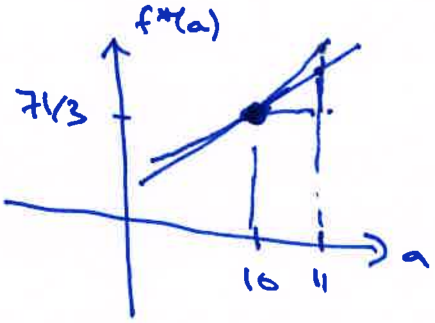
c) Estimate: max $16 - x^4 - 2x^2 - 3y^2 + 6xz - 6z^2 + 11z$

$$\boxed{\max f(x,y,z;a) = 16 - x^4 - 2x^2 - 3y^2 + 6xz - 6z^2 + az}$$

unconstrained max problem with parameter a

a=10: $f^*(10) = 71/3$ ← max value for $a=10$

$$\left.\begin{aligned} x^*(10) &= 1 \\ y^*(10) &= 0 \\ z^*(10) &= 4/3 \end{aligned}\right\} \leftarrow \text{max pt} \quad \text{— " —}$$



$$f^*(11) \simeq \underset{\substack{\| \\ 71/3}}{f^*(10)} + \underset{\substack{\| \\ 11-10}}{\Delta a} \cdot \underset{\substack{\| \\ 4/3}}{\frac{df^*(a)}{da}} = 25$$

Env. thm:

$$\frac{df^*(a)}{da} = f'_a\left(x^*(a), y^*(a), z^*(a)\right) = z^*(a) = 4/3$$

(a=10)

$f'_a = z$

4. max $f = 3x^2 - y^2 - 2z^2$ when $2x^4 + 2y^4 + z^4 \le 18$

Bounded set:
$x^4 \le 18/2 = 9$
$y^4 \le 9$
$z^4 \le 18$

KT problem in std. form: max + $\le$ ✓

a) $\mathcal{L} = 3x^2 - y^2 - 2z^2 - \lambda(2x^4 + 2y^4 + z^4)$

FOC:
$$\mathcal{L}'_x = 6x - \lambda \cdot 8x^3 = 0$$
$$\mathcal{L}'_y = -2y - \lambda \cdot 8y^3 = 0$$
$$\mathcal{L}'_z = -4z - \lambda \cdot 4z^3 = 0$$

C: $2x^4 + 2y^4 + z^4 \le 18$

CSC: $\lambda \ge 0$ and $\lambda \cdot (2x^4 + 2y^4 + z^4 - 18) = 0$

Alt: CSC: $\lambda \ge 0$, and $\lambda = 0$ if $2x^4 + 2y^4 + z^4 < 18$

Alt: C+CSC:
$$\begin{cases} 2x^4 + 2y^4 + z^4 = 18, & \lambda \ge 0 \\ \text{or} & \\ 2x^4 + 2y^4 + z^4 < 18, & \lambda = 0 \end{cases}$$

b)

	Binding	Non-binding
C	$2x^4 + 2y^4 + z^4 = 18$	$2x^4 + 2y^4 + z^4 < 18$
CSC	$\lambda \ge 0$	$\lambda = 0$
FOC	$6x - 8\lambda x^3 = 0$ $-2y - 8\lambda y^3 = 0$ $-4z - 4\lambda z^3 = 0$	$6x - \cancel{8\lambda x^3} = 0$ $-2y - \cancel{8\lambda y^3} = 0$ $-4z - \cancel{4\lambda z^3} = 0$
	$x(6 - 8\lambda x^2) = 0$ $y(-2 - 8\lambda y^2) = 0$ $z(-4 - 4\lambda z^2) = 0$	$x = 0$, $y = 0$, $z = 0$, $\lambda = 0$ $0 < 18$ ok $(0,0,0;0)$ cand. pt $f(0,0,0) = 0$

Binding: (ctd.)

$x = 0$ or $6 - 8\lambda x^2 = 0$

$y = 0$ or $\cancel{-2 - 8\lambda y^2 = 0}$

$z = 0$ or $\cancel{-4 - 4\lambda z^2 = 0}$

not possible since $\lambda \geq 0$, $x^2, y^2, z^2 \geq 0$

$\Downarrow$

$$\boxed{\begin{array}{l}\lambda \geq 0 \text{ ok} \\ 2x^4 + 2y^4 + z^4 = 18\end{array}}$$

FOC $\begin{cases} y = 0 \\ z = 0 \\ x = 0 \text{ or } \lambda = \dfrac{6}{8x^2} \end{cases}$

i) $x = 0, y = 0, z = 0$: $2x^4 + 2y^4 + z^4 = 0 \neq 18$ no cand pts.

ii) $\lambda = \dfrac{6}{8x^2}, y = 0, z = 0$: $\lambda \geq 0$ ok

$2x^4 + 0 + 0 = 18$

$x^4 = 9$

$x^2 = 3$

$x = \pm\sqrt{3}$

$\lambda = \dfrac{6}{8 \cdot 3} = \dfrac{6}{24} = 1/4$

Cand.pt: $(x, y, z; \lambda) = (\pm\sqrt{3}, 0, 0; 1/4)$

$f = 9$

$$\boxed{\begin{array}{lll}\text{Cand: } (x, y, z; \lambda) = & (0, 0, 0; 0), & f = 0 \\ (= \text{pts that satisfy KT cond.}) & (\pm\sqrt{3}, 0, 0; 1/4) & f = 9\end{array}}$$

c) The set of adm. pts. $\{(x, y, z) : 2x^4 + 2y^4 + z^4 \leq 18\}$ is bounded $\Big\}$ $\xRightarrow{\text{EVT}}$ there is a max

either regular cand. pt

or

adm. pts where NDCQ fails

NDCQ: $2x^4+2y^4+z^4 \le 18$

Binding: $2x^4+2y^4+z^4=18$ $\quad \text{rk}\begin{pmatrix}8x^3 & 8y^3 & 4z^3\end{pmatrix}=1$

NDCQ fails: $8x^3=0 \quad 8y^3=0 \quad z^3=0$

$\Downarrow$

$(x,y,z)=(0,0,0)$ not adm.

Nonbinding: no condition

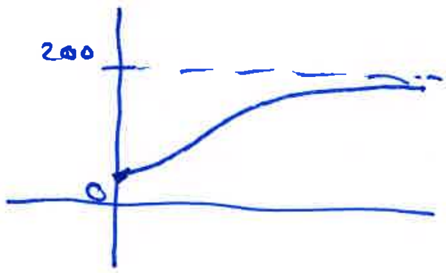
NDCQ satisfied at all adm. pts.

$\Downarrow$

$f_{max} = 9$ at $(x,y,z;\lambda) = (\pm\sqrt{3},0,0;1/4)$

5. $y' = 0.15y\,(1-y/200)$

$= \underbrace{0.15}_{f(t)} \cdot \underbrace{y\,(1-y/200)}_{g(y)}$



$\dfrac{1}{y\,(1-y/200)}\, y' = 0.15$

$\dfrac{200}{y\,(200-y)}\, y' = 0.15 \quad \Rightarrow \quad \displaystyle\int \frac{200}{y\,(200-y)}\,dy = \int 0.15\,dt$

$\Downarrow$

$\dfrac{A}{y} + \dfrac{B}{200-y}$

$$\int \frac{200}{y(200-y)}\,dy = \int 0.15\,dt$$

Partial fractions:

$$\frac{200}{y(200-y)} = \frac{A}{y} + \frac{B}{200-y} \quad \Big|\cdot y(200-y)$$

$$200 = A(200-y) + By$$

$$= 200A + y(B-A)$$

$$B-A=0$$

$$200A = 200 \Rightarrow \underline{A=1,\ B=1}$$

$$\int \frac{200}{y(200-y)}\,dy$$

$$= \int \frac{1}{y} - \frac{1}{200-y}\,dy$$

$$= \ln|y| - \ln|200-y| + C_1$$

$$\int 0.15\,dt = 0.15t + C_2$$

$$\ln|y| - \ln|200-y| = 0.15t + C_2 - C_1 \quad \Big|\, e^{*}$$

$$e^{\ln|y| - \ln|200-y|} = e^{0.15t + C_2 - C_1}$$

$$\frac{|y|}{|200-y|} = e^{0.15t} \cdot e^{C_2-C_1}$$

$$\frac{y}{200-y} = \pm e^{C_2-C_1} e^{0.15t} = C\,e^{0.15t} \qquad \left(C = \pm e^{C_2-C_1}\right)$$

$$y = (200-y)\left(Ce^{0.15t}\right)$$

$$y\left(1 + Ce^{0.15t}\right) = 200\cdot Ce^{0.15t}$$

$$y = 200\cdot\frac{Ce^{0.15t}}{1 + C\cdot e^{0.15t}}$$

$y_0 = 50$: $\quad 50 = 200 \cdot \dfrac{Ce^0}{1+Ce^0} = 200 \cdot \dfrac{C}{1+C}$

$$50(1+C) = 200C$$
$$50 = 150C$$
$$C = 50/150 = 1/3$$

$$y(t) = 200 \cdot \frac{1/3\, e^{0.15t}}{1 + 1/3 \cdot e^{0.15t}}$$

$\underline{K=200}$ carrying capacity $\quad (y(t) \to 200$ as $t \to \infty)$

90%: $\quad y = 200 \cdot 0.90 = 200 \cdot \dfrac{1/3\, e^{0.15t}}{1 + 1/3 \cdot e^{0.15t}}$

$$\Downarrow$$

$$\frac{1/3\, e^{0.15t}}{1 + 1/3\, e^{0.15t}} = 0.90$$

$$1/3\, e^{0.15t} = 0.90\left(1 + \tfrac{1}{3} e^{0.15t}\right)$$

$$\left(\tfrac{1}{3} - 0.90 \cdot \tfrac{1}{3}\right) e^{0.15t} = 0.90$$

$$e^{0.15t} = \frac{0.90}{0.10 \cdot 1/3} = \frac{9}{1/3} = 27$$

$$0.15t = \ln(27) = \ln(3^3) = 3\ln 3$$

$$t = \frac{3\ln 3}{0.15} = \underline{\underline{20 \ln 3}}$$

Exam result

GRA60353 Mathematics, 27/11/2018

Statistics

Grade A-B	41.4%
Grade F	9.6%
Average score	69.9% (C)

Comments

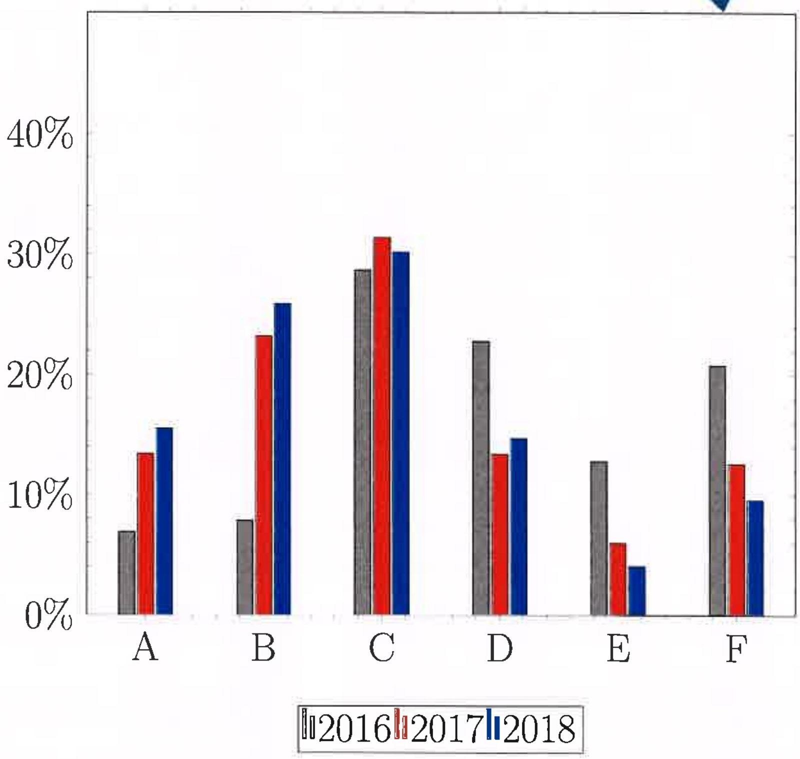
The results on the final exam were very good. A summary of average score per problem is shown below. In Question 1(b), the scores were a bit lower than expected, as many did not use the correct definition of a nullspace, and quite few knew the correct definition of dimension of the nullspace. Otherwise, the scores were very good. As expected, the scores were higher for the (a) problems, and lower for the (c) problems. Especially for Question 2(c), 3(bc) and 4(c), which do not require difficult computations, the score depends more on clear, concise arguments with reference to relevant theory, and less on correct answers.

Average score per problem

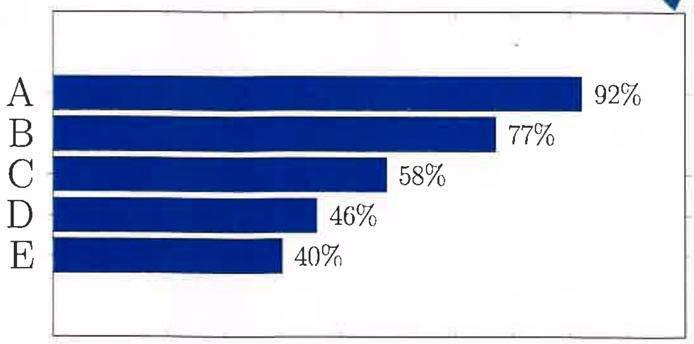
- Question 1: a) 92% b) 45% c) 65%
- Question 2: a) 89% b) 72% c) 63%
- Question 3: a) 91% b) 56% c) 66%
- Question 4: a) 95% b) 53% c) 43%
- Question 5: 11% (extra credit)

Detailed solutions to the exam problems can be found at www.dr-eriksen.no.

Grade distribution



Grading scale



GRA 6035 MATHEMATICS

Common mistakes: Final exam 11/2018

- not knowing the meaning of terms, especially:

  Null(A) = all solutions of $A \cdot \underline{x} = \underline{0}$

  dim Null(A) = # free variables in $A \cdot \underline{x} = \underline{0}$

- using RRC (reduced rank condition) without checking the assumptions:

  $rk\,A = \underline{3}$ and $D_1, D_2, \underline{D_3} > 0 \Rightarrow A$ is positive semi-definite

  (and $D_4 = 0$ follows from the assumptions)

- more generally, not giving reasons for conclusions and not refering to theory, especcially:

  * not saying why A is positive semidefinite (refer to RRC if it applies, or mention principal minors)
  * not saying why equilibrium states are stable / unstable and especially globally asymptotically stable
  * not refering to the envelope theorem correctly and not giving reasons why the slope of the tangent line of $f^*(a)$ has its value
  * not writing arguments clearly enough (especially important to get an A)

- when showing that f has a global max in 3b, to use the Hessian at the stationary point and not the Hessian at a general point, and to not argue well enough for the fact that $D_3 < 0$ at all points

- not understanding the notation $f^*(a)$, $x^*(a)$, etc and not writing the envelope theorem in a correct and meaningful way:

$$\frac{df^*(a)}{da} = f'_a\left(x^*(a), y^*(a), z^*(a)\right)$$

derivative of $f^*(a)$, i.e. the slope of its tangent line at the given point, where $f^*(a)$ is maximum value function

derivative of f, the fn. given in the problem, with respect to the parameter a, when we replace

$x \longmapsto x^*(a)$

$y \longmapsto y^*(a)$

$z \longmapsto z^*(a)$

To keep in mind for the retake exam:

- give reason for your answers, even when you think it should not be necessary
- show that there is a clear argument from the given information and theory in the course, to the answer
- make sure you know the theory, especially definitions such as Null(A) - check lecture notes
- answer all questions
- use your time well on the exam, and make sure you have enough time for all 12 points

The comments above, about giving reason for answers, does not mean that you should write long answers - arguments are best when they are short and to the point