Plan:

1. Markov chains and diagonalization
2. Quadratic forms and definiteness

Review:

a) Eigenvalues and eigenvectors

A $n \times n$ matrix

$A \cdot \underline{v} = \lambda \underline{v}$

$\Updownarrow$

$(A - \lambda I)\underline{v} = \underline{0}$ ← homogeneous lin. system with parameter λ

Eigenvalues:

$|A - \lambda I| = 0$ ← Solutions = eigenvalues

char. eqn. (polynomial eqn. of degree n)

Multiplicity: $\lambda = \lambda^*$ has multiplicity m

$\Updownarrow$

$(\lambda - \lambda^*)^m$ is a factor in $|A - \lambda I| = 0$

Eigenvectors:

$E_\lambda = \text{Null}(A - \lambda I)$ ← all sol. of $(A - \lambda I) \cdot \underline{v} = \underline{0}$

$\boxed{\dim E_\lambda = \# \text{ free var's}}$

Result: If λ is an eigenvalue of multiplicity m, then

$$\boxed{1 \leq \dim E_\lambda \leq m}$$

$\uparrow$ # free var's

b) Diagonalization:

$\underset{n\times n}{A}$ A diagonalizable: $P^{-1}AP = D$

P: invertible matrix; D: diagonal matrix

Result:

A diag. $\iff$
i) there are n eigenvalues $\lambda_1, \dots, \lambda_n$ (counted with multiplicity)
ii) there are n linearly independent eigenvectors $\underline{v_1}, \dots, \underline{v_n}$

$$D = \begin{pmatrix} \lambda_1 & & & 0 \\ & \lambda_2 & & \\ & & \ddots & \\ 0 & & & \lambda_n \end{pmatrix}$$

$$P = (\underline{v_1} | \underline{v_2} | \cdots | \underline{v_n})$$

$$A \cdot \underline{v_i} = \lambda_i \underline{v_i}$$

Useful results:

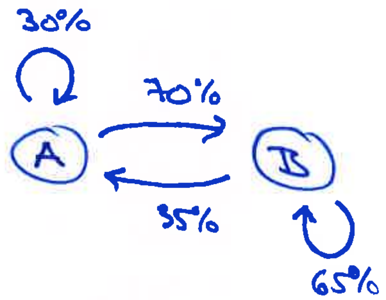
i) A symmetric $\Rightarrow$ A diagonalizable

ii) In general: A diag. $\iff$
i) there are n eigenval. (counted with mult.)
ii) for any eigenvalue λ of mult. $m > 1$, we have $\boxed{\dim E_\lambda = m}$
(# free vars = mult.)

① Markov chains

Ex: Two companies A and B share a market.

Marketshares: A: 88%
B: 12%



(state vector)

$\underline{v}_0 = \begin{pmatrix} 0.88 \\ 0.12 \end{pmatrix}$ market share vector at time $t=0$

$$A = \begin{pmatrix} 0.30 & 0.35 \\ 0.70 & 0.65 \end{pmatrix}$$

transition matrix

$$\underline{v}_1 = A \cdot \underline{v}_0 = \begin{pmatrix} 0.30 & 0.35 \\ 0.70 & 0.65 \end{pmatrix} \cdot \begin{pmatrix} 0.88 \\ 0.12 \end{pmatrix}$$

$$= \begin{pmatrix} 0.30 \cdot 0.88 + 0.35 \cdot 0.12 \\ 0.70 \cdot 0.88 + 0.65 \cdot 0.12 \end{pmatrix} = \begin{pmatrix} 0.306 \\ 0.694 \end{pmatrix}$$

$$\underline{v}_2 = A \underline{v}_1 = A \cdot A \cdot \underline{v}_0 = A^2 \cdot \underline{v}_0 = \begin{pmatrix} 0.3347 \\ 0.6653 \end{pmatrix}$$

$\vdots$

$$\boxed{\underline{v}_n = A^n \cdot \underline{v}_0}$$

What happens when n is big, $n \to \infty$

We find eigenvalues and eigenvectors of A:

$$\begin{vmatrix} 0.30-\lambda & 0.35 \\ 0.70 & 0.65-\lambda \end{vmatrix} = 0 \qquad \lambda^2 - 0.95\lambda - 0.05 = 0$$

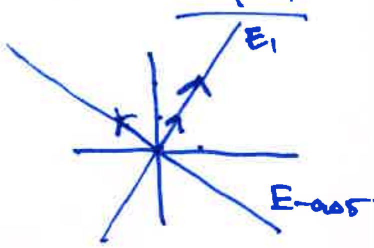
$$\underline{\lambda_1 = 1}, \quad \lambda_2 = \underline{-0.05}$$

E_1: $\begin{pmatrix} -0.70 & 0.35 \\ \cancel{0.70} & \cancel{-0.35} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\quad \begin{pmatrix} x \\ y \end{pmatrix} = \underline{t \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$ $\quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y/2 \\ y \end{pmatrix} = y \cdot \underline{\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}}$

$$-0.7x + 0.35y = 0 \Rightarrow x = y/2$$

$E_{-0.05}$: $\begin{pmatrix} 0.35 & 0.35 \\ \cancel{0.70} & \cancel{0.70} \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$$\begin{pmatrix} x \\ y \end{pmatrix} = t \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$



$\boxed{\text{BREAK}}$

Method I: Diagonalization

P.1 $\quad P^{-1}AP = D \quad$ with $\quad P = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \qquad D = \begin{pmatrix} 1 & 0 \\ 0 & -0.05 \end{pmatrix}$

$$\Downarrow$$

$$AP = PD \quad | \cdot P^{-1}$$

$$\Downarrow$$

$$\underline{A = PDP^{-1}}$$

$$A^n = (PD\cancel{P^{-1}}) \cdot (\cancel{P}D\cancel{P^{-1}}) \cdots (\cancel{P}DP^{-1}) = PD^nP^{-1}$$

$$A^n = P \cdot D^n \cdot P^{-1} = \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1^n & 0 \\ 0 & (-0.05)^n \end{pmatrix} \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

$$(n \to \infty) \longrightarrow \begin{pmatrix} 1 & -1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \cdot \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 1 & 0 \\ 2 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 1 \\ -2 & 1 \end{pmatrix} = \underline{\underline{\frac{1}{3} \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}}}$$

Equilibrium state:

$$\underline{v_n} = A^n \cdot \underline{v_0} \quad \Rightarrow \quad \underline{v} = \lim_{n \to \infty} \underline{v_n} = \begin{pmatrix} 1/3 & 1/3 \\ 2/3 & 2/3 \end{pmatrix} \begin{pmatrix} 0.88 \\ 0.12 \end{pmatrix} = \underline{\underline{\begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}}}$$

$$\boxed{\begin{array}{l} \text{Marked shares:} \\ A: 1/3 = 33.3\% \\ B: 2/3 = 66.7\% \end{array}}$$

Method 2: General results for Markov chains

a_{ij}: transition from node j to node i

General setup:

State vector: $\underline{v} = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

$v_i \geq 0$

$v_1 + v_2 + \dots + v_n = 1$

Transition matrix:

$$A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}$$

$a_{ij} \geq 0$

column sums $= 1$

<u>Equations:</u> $\underline{v}_{i+1} = A \cdot \underline{v}_i$ $\qquad \underline{v}_m = A^m \cdot \underline{v}_0$

<u>Results:</u>

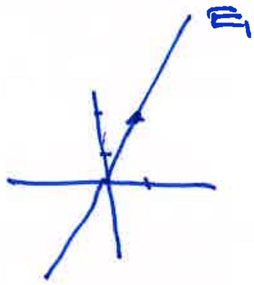
Assume that the markov chain is <u>regular</u>. Then:

i) $\lambda = 1$ is an eigenvalue, and there is a unique vector $\underline{v}$ in E_1 that is a state vector.

ii) $\lim_{n \to \infty} A^n \cdot \underline{v}_0 = \underline{v}$ is the <u>equilibrium state</u>

<u>Defn:</u> A Markov chain is <u>regular</u> if A^n consists of strictly positive entries for n large enough.

All $a_{ij} > 0 \implies A$ is regular



<u>Ex:</u> $A = \begin{pmatrix} 0.30 & 0.35 \\ 0.70 & 0.65 \end{pmatrix}$

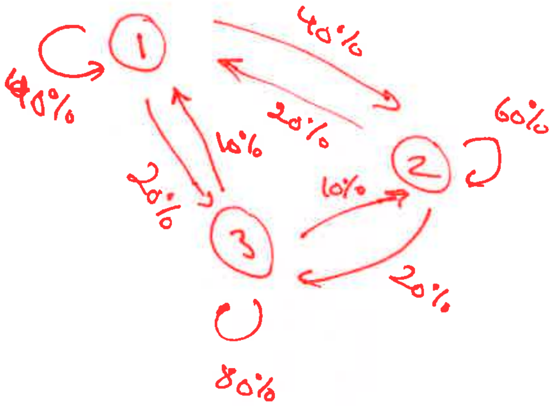
<u>$\lambda = 1$:</u> E_1: $\begin{pmatrix} -0.70 & 0.35 \\ 0.70 & -0.35 \end{pmatrix} \longrightarrow \underline{t \cdot \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$

$x = t$
$y = 2t$
$x + y = 3t = 1$
$t = 1/3$

$\underline{v} = \underline{\underline{\begin{pmatrix} 1/3 \\ 2/3 \end{pmatrix}}}$

Ex: Markov chain with transition matrix:
$$A = \begin{pmatrix} 0.4 & 0.2 & 0.1 \\ 0.4 & 0.6 & 0.1 \\ 0.2 & 0.2 & 0.8 \end{pmatrix}$$

A is regular since all entries in A are >0, i.e. no zero entries



$\lambda = 1$:
$$\begin{pmatrix} -0.6 & 0.2 & 0.1 \\ 0.4 & -0.4 & 0.1 \\ 0.2 & 0.2 & -0.2 \end{pmatrix} \begin{array}{l} \cdot 10 \\ \cdot 10 \\ \cdot 10 \end{array}$$

$$\begin{pmatrix} 2 & 2 & -2 \\ 4 & -4 & 1 \\ -6 & 2 & 1 \end{pmatrix} \begin{array}{l} \\ -2 \\ 3 \end{array} \to \begin{pmatrix} 2 & 2 & -2 \\ 0 & -8 & 5 \\ \cancel{0} & \cancel{8} & \cancel{-5} \end{pmatrix}$$

$$2x + 2y - 2z = 0$$
$$-8y + 5z = 0$$
z free

$x = -y + z = -\frac{5}{8}z + z = \underline{\frac{3}{8}z}$

$y = \underline{5z/8}$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 3z/8 \\ 5z/8 \\ z \end{pmatrix} = z/8 \cdot \begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}$$

State vectors:

$3+5+8=16$

$\Downarrow$

$$\underline{v} = \frac{1}{16}\begin{pmatrix} 3 \\ 5 \\ 8 \end{pmatrix}$$

By the result for regular Markov chains, the equilibrium state is
$$\begin{pmatrix} 3/16 \\ 5/16 \\ 8/16 \end{pmatrix} = \begin{pmatrix} 0.1875 \\ 0.3125 \\ 0.50 \end{pmatrix}$$

(2) Definiteness of quadratic forms

Ex: $f(x_1,x_2,x_3) = x_1^2 - x_2^2 + 2x_2x_3 + 7x_3^2$ is a quadratic form in 3 variables

Defn: A quadratic form is a function $f(x_1,\ldots,x_n)$ in n variables which is polynomial such that each term has degree two.

n=1: $f(x) = ax^2$

n=2: $f(x,y) = ax^2 + bxy + cy^2$

n=3: $f(x,y,z) = ax^2 + bxy + cxz + dy^2 + eyz + fz^2$

Matrix form:

Ex: $f(x_1,x_2,x_3) = x_1^2 - x_2^2 + 2x_2x_3 + 7x_3^2$

$\underline{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$: $f(\underline{x}) = (x_1 \; x_2 \; x_3) \cdot \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 1 \\ 0 & 1 & 7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \underline{x}^T \cdot A \cdot \underline{x}$

For each quadr. form $f(\underline{x})$, there is a unique symmetric matrix A s.t.

$$f(\underline{x}) = \underline{x}^T \cdot A \cdot \underline{x}$$

$\boxed{\text{BREAK}}$

Ex: $f(x_1,x_2,x_3) = \underline{2x_1^2} - x_1x_2 + 4x_1x_3 + \underline{7x_2^2} - 4x_2x_3 + \underline{x_3^2}$

$$f(\underline{x}) = \underline{x}^T A \underline{x} = (x_1\ x_2\ x_3) \cdot \begin{pmatrix} 2 & -1/2 & 2 \\ -1/2 & 7 & -2 \\ 2 & -2 & 1 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$x_1^2 = x_1 \cdot x_1 \rightarrow (1,1)$ $\qquad x_1 \cdot x_2 \rightarrow (1,2)$, $x_2 \cdot x_1 \rightarrow (2,1)$

$x_2^2 = x_2 \cdot x_2 \rightarrow (2,2)$

$x_3^2 = x_3 \cdot x_3 \rightarrow (3,3)$ $\qquad \left.\begin{matrix} x_1 \cdot x_3 \\ x_3 \cdot x_1 \end{matrix}\right\} (1,3),(3,1)$

In general: $\underset{n \times n}{A}$ symmetric matrix of $\underset{n \text{ var.}}{f}$

$$A \rightleftarrows f$$

Definiteness:

$f = f(x_1,\dots,x_n)$ quadratic form

Def: f is called positive semi-definite if $f(\underline{x}) \geq 0$ for all $\underline{x}$

— " — positive definite " $f(\underline{x}) > 0$ for $\underline{x} \neq \underline{0}$

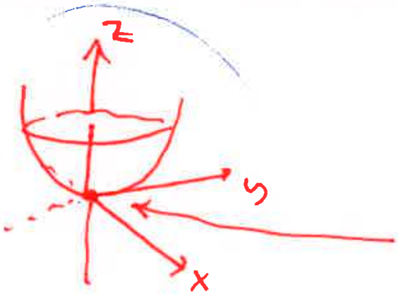
main cases — " — negative semidefinite " $f(\underline{x}) \leq 0$ for all $\underline{x}$

— " — negative definite " $f(\underline{x}) < 0$ for $\underline{x} \neq \underline{0}$

— " — indefinite otherwise (has both pos. and neg. values)

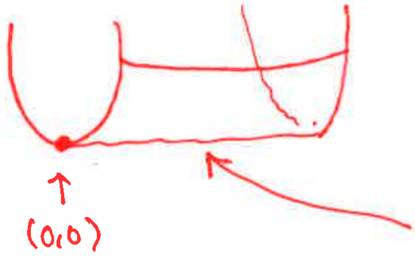
Definiteness: Interpretation in terms of the graph of the quadratic form f

f pos. defn. :



global min: $(0,0)$

f pos. semidefn. :

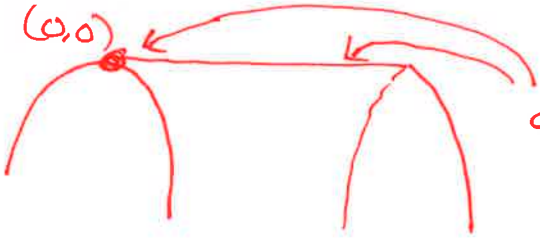


line of global min's including $(0,0)$

f neg. defn :

global max $(0,0)$

f neg. Semidefn:



a line of global max incl. $(0,0)$

f indefinite : $(0,0)$ is saddle point.

Note: For all quadr. forms:

* $f(0,0,\ldots,0)=0$
* $(0,0,\ldots,0)$ is a stationary pt

Methods for classifying quadratic forms:

$f(x_1, \ldots, x_n)$ quadr. form

$\downarrow$

A s.t. $f(\underline{x}) = \underline{x}^T A \underline{x}$

$n \times n$ symm. matrix

classify = determine the definiteness of f

A symmetric

$\Downarrow$

A has n eigenvalues (counted with mult.):

Method I: Eigenvalues of A : $\lambda_1, \ldots, \lambda_n$

A pos. definite $\Leftrightarrow$ $\lambda_1, \ldots, \lambda_n > 0$

A pos. semi defin. $\Leftrightarrow$ $\lambda_1, \ldots, \lambda_n \geq 0$

A neg. defin. $\Leftrightarrow$ $\lambda_1, \ldots, \lambda_n < 0$

A neg. semi. defin. $\Leftrightarrow$ $\lambda_1, \ldots, \lambda_n \leq 0$

A indefinite $\Leftrightarrow$ there are both pos. and neg. eigenvalues

Explanation: Any quadratic form can be written

$$f(x_1 .., x_n) = \lambda_1 \cdot u_1^2 + \lambda_2 u_2^2 + \ldots + \lambda_n u_n^2$$

quadr. form without cross-terms

where $\lambda_1, .., \lambda_n$: eigenvalues of A
$u_1, \ldots, u_n$: expressions (linear) in $x_1 .., x_n$ given by eigenvectors of A

Ex: $f(x,y) = 2xy$

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\lambda^2 - 1 = 0$$

$$\lambda = \pm 1$$

E_1: $y \cdot \begin{pmatrix} 1 \\ 1 \end{pmatrix}$

E_{-1}: $y \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$f(x,y) = 2xy$$
$$= \tfrac{1}{2}(x+y)^2 - \tfrac{1}{2}(x-y)^2$$
$$= 1 \cdot \left(\frac{x+y}{\sqrt{2}}\right)^2 - 1 \cdot \left(\frac{x-y}{\sqrt{2}}\right)^2$$
$$= \underline{\underset{\uparrow \lambda_1}{1} \cdot u_1^2 - \underset{\uparrow \lambda_2}{1} \cdot u_2^2}$$

$$u_1 = \frac{x+y}{\sqrt{2}} \qquad u_2 = \frac{x-y}{\sqrt{2}}$$

Method 2: Principal minors (leading principal minors)

Leading principal minors:

Ex: $A = \begin{pmatrix} 2 & -1/2 & 2 \\ -1/2 & 7 & -2 \\ 2 & -2 & 1 \end{pmatrix}$

$D_1 = 2$

$D_2 = \begin{vmatrix} 2 & -1/2 \\ -1/2 & 7 \end{vmatrix} = 14 - \frac{1}{4} > 0$

$D_3 = |A| = \begin{vmatrix} 2 & -1/2 & 2 \\ -1/2 & 7 & -2 \\ 2 & -2 & 1 \end{vmatrix}$

$= 2 \cdot (1-14) + 2(-4+1) + 1 \cdot (14 - 1/4)$

$= -26 - 6 + 13.75 < 0$

$\boxed{\begin{matrix} D_1 > 0 \\ D_2 > 0 \\ D_3 < 0 \end{matrix}}$ indefinite

Result:

$D_1, D_2, \ldots, D_n > 0 \iff A$ positive defn.

$D_1 < 0,\ D_2 > 0, \ldots$
$\Updownarrow$
$(-1)^i \cdot D_i > 0$ for all i
$\iff A$ negative defn.

A positive semidefn. $\Rightarrow D_1 \geq 0, D_2 \geq 0, \ldots, D_n \geq 0$

A negative semidefn. $\Rightarrow D_1 \leq 0,\ D_2 \geq 0, \ldots, (-1)^n \cdot D_n \geq 0$

Leading principal minor:

$D_i = M_{123\ldots i,\,123\ldots i}$ ← keep rows $1, 2, \ldots, i$ cols $1, 2, \ldots, i$

first i rows, first i cols.

Explanation:

$$A = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{pmatrix}$$

$\lambda_1 = -1$
$\lambda_2 = -2$
$\lambda_3 = -3$

$D_1 = \lambda_1 = -1 < 0$
$D_2 = \lambda_1 \cdot \lambda_2 = 2 > 0$
$D_3 = \lambda_1 \cdot \lambda_2 \cdot \lambda_3 = -6 < 0$

neg. def.

Result:

A positive semidef. $\Longleftrightarrow$ $\Delta_i \geq 0$ for all principal i-minors Δ_i

A negative semidef. $\Longleftrightarrow$ $\Delta_1 \leq 0, \Delta_2 \geq 0, \ldots$ for all principal minors Δ_i of order i

Ex: $\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} \boxed{1} & 2 \\ 2 & \boxed{4} \end{pmatrix}$ $D_1 = 1$, $D_2 = 0$ $\Delta_1 = 1, 4$, $\Delta_2 = 0$

pos. semidef.

$\Delta_1 : M_{1,1} \quad M_{2,2}$

keep same row- and col- numbers

$\Rightarrow$ "symmetric minor"

Principal minors:

$\Delta_k : M_{i_1 i_2 \ldots i_k, i_1 i_2, \ldots, i_k}$
"
$M_{i_1 i_2 \ldots i_k, i_1 i_2 \ldots i_k}$

keep same rows $i_1, i_2, \ldots, i_k$
and same cols $i_1, i_2, \ldots, i_k$

Ex: $f = x^2 + y^2 + z^2 - 2xz$

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

Leading principal minors:

$D_1 = 1$

$D_2 = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$

$D_3 = 1 \cdot \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = 1 \cdot 0 = 0$

$D_1 > 0$
$D_2 > 0$
$D_3 = 0$

can be positive semi-definite, but must check all principal minors since $D_3 = 0$.

$\Delta_1 = M_{1,1}, M_{2,2}, M_{3,3}$

$= \underline{1}, \underline{1}, \underline{1}$

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$\Delta_2 = M_{12,12}, M_{23,23}, M_{13,13}$

$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}, \begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix}$

$= \underline{1}, \underline{1}, \underline{0}$

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}$$

$$\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

$\Delta_3 = \begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix} = M_{123,123}$

$= 0$

Concl: All $\Delta_1, \Delta_2, \Delta_3 \geq 0 \Rightarrow A$ positive semidefinite

Ex: $f = x^2 + 4xy + 8xz + 3y^2 - 2yz + 2z^2$

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 2 & 3 & -1 \\ 4 & -1 & 2 \end{pmatrix}$$

$D_1 = 1 > 0$

$D_2 = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} = 3 - 4 = -1 < 0$

A indefinite

($D_2 \geq 0$ for both pos. semidefn. and neg. semidefn.)