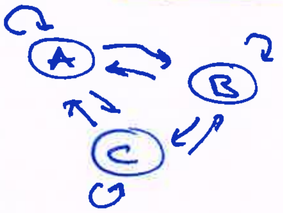
## Plan:

1. Definiteness: Reduced rank criterion
2. Unconstrained optimization
3. Convex and concave functions

## Review:

(a) Markov chains:




$\longrightarrow A = (a_{ij})$ transition matrix

$a_{ij}$ = transitions from node $j$ to node $i$

$A \cdot \underline{v}_n = \underline{v}_{n+1}$ ← state vector

$\underline{v}_0$: starting state

**Methods:**

i) A is regular:

$\underline{v}$ is the unique eigenvector with $\lambda = 1$ that is a state vector.

**Equilibrium state:**

$$\underline{v} = \lim_{n\to\infty} \underbrace{A^n \cdot \underline{v}_0}_{\underline{v}_n}$$

A regular $\iff$ $A^n$ consists of $>0$ entries for $n$ large enough $\iff$ there is a path from any node to any other node

$$\boxed{\left.\begin{matrix} a_{ij} > 0 \text{ for} \\ \text{all } i,j \end{matrix}\right\} \Rightarrow A \text{ is regular}}$$

ii) The general case: Direct computation

Compute $A^n$ Use: $A^n = PD^nP^{-1}$

## b) Definiteness of quadratic forms

$f(x_1, \ldots, x_n) = \underbrace{a_{11} \cdot x_1^2 + \ldots}_{\text{terms of degree 2}}$

= quadratic form

$\Rightarrow f(\underline{x}) = \underline{x}^T \cdot A \cdot \underline{x}$

for a symmetric $n \times n$-matrix $A$

$\boxed{\text{Eigenvalues of } A: \; \lambda_1, \lambda_2, \ldots, \lambda_n}$

Hard to compute $\lambda_1, \ldots, \lambda_n$!

### Definiteness:

$f$ positive semi-defn: $f(\underline{x}) \geq 0$ for all $\underline{x}$ $\Leftrightarrow \lambda_1, \ldots, \lambda_n \geq 0$

$f$ negative — " —: $f(\underline{x}) \leq 0$ for all $\underline{x}$ $\Leftrightarrow \lambda_1, \ldots, \lambda_n \leq 0$

$f$ indefinite: $f$ has both pos. and neg. values $\Leftrightarrow$ there are both pos. and neg. eigenvalues

$f$ pos. defn.: $f(\underline{x}) > 0$ for $\underline{x} \neq 0$ $\Leftrightarrow \lambda_1, \ldots, \lambda_n > 0$

$f$ neg. defn: $f(\underline{x}) < 0$ for $\underline{x} \neq \underline{0}$ $\Leftrightarrow \lambda_1, \lambda_2, \ldots, \lambda_n < 0$

### Method using principal minors:

A leading principal minor: $D_1, D_2, D_3, \ldots, D_n$

$D_i$ leading principal minor of order $i$: keep the first $i$ rows, — " — $i$ cols

A principal minor: $\Delta_1, \Delta_2, \Delta_3, \ldots, \Delta_n$

$\Delta_i$ principal minor of order $i$: keep $i$ rows and the same $i$ cols

Results: $D_1, D_2, \ldots, D_n > 0 \iff A$ pos. defn.
$D_1, D_2 \ldots, D_n \geq 0 \impliedby A$ pos. semidefn
$\Delta_1, \Delta_2, \ldots, \Delta_n \geq 0 \iff A$ pos. semidefn.

Ex: $A = \begin{pmatrix} 4 & 2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}$

$D_1 = \underline{4}$
$D_2 = 4 - 4 = \underline{0}$
$D_3 = |A| = 3 \cdot 0 = \underline{0}$

$M_{12,12}$ $M_{23,23}$, $M_{13,13}$

$\Delta_1 = 4, 1, 3 \geq 0$
$\Delta_2 = 0, 3, 12 \geq 0$
$\Delta_3 = 0$

$\Bigg\}$ A pos. semidefn.

Results. $D_1 < 0, D_2 > 0, \ldots, (-1)^n D_n > 0 \iff A$ neg. defn.
$D_1 \leq 0, D_2 \geq 0, \ldots, (-1)^n D_n \geq 0 \impliedby A$ neg. semidefn.
$\Delta_1 \leq 0, \Delta_2 \geq 0, \ldots, (-1)^n \Delta_n \geq 0 \iff A$ neg. semi-defn.

Typical indefinite examples: $\Delta_2 < 0$, $\Delta_4 < 0, \ldots$
$\Delta_1$ and $\Delta_3$ have opposite signs

Ex: $A = \begin{pmatrix} 4 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{pmatrix}$

$D_1 = 4$
$D_2 = 0$
$D_3 = 2 \cdot 0 - 1 \cdot 0 = 0$

$\Delta_1 = 4, 1, 0$
$\Delta_2 = 0, -1, -4$
$\Delta_3 = 0$

$\uparrow$
neg. $\Delta_2$

A indefinite

## Extra example:

$$f(x,y,z,w) = x^2 + y^2 + 6yz - 4yw + 9z^2 - 12zw + 4w^2$$

$$A = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 3 & 9 & -6 \\ 0 & -2 & -6 & 4 \end{pmatrix}$$

$D_1 = \underline{1}$

$D_2 = \underline{1}$

$D_3 = 1 \cdot \begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 1 \cdot (9-9) = \underline{0}$

$D_4 = 1 \cdot \begin{vmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ -2 & -6 & 4 \end{vmatrix} = 1 \cdot \big(1 \cdot (36-36) - 3 \cdot (12-12) + (-2) \cdot (-18+18)\big)$

$= \underline{0}$

pos. semidef. ?

$$\Delta_1 = \underbrace{1}_{D_1 = M_{1,1}},\ \underbrace{1}_{M_{2,2}},\ \underbrace{9}_{M_{3,3}},\ \underbrace{4}_{M_{4,4}} \geq 0 \quad \checkmark \text{ ok.}$$

$$\Delta_2 = \underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1}_{D_2 = M_{12,12}},\ \underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 9 \end{vmatrix} = 9}_{M_{13,13}},\ \underbrace{\begin{vmatrix} 1 & 0 \\ 0 & 4 \end{vmatrix} = 4}_{M_{14,14}},\ \underbrace{\begin{vmatrix} 1 & 3 \\ 3 & 9 \end{vmatrix} = 0}_{M_{23,23}},\ \underbrace{\begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} = 0}_{M_{24,24}},$$

$$\underbrace{\begin{vmatrix} 9 & -6 \\ -6 & 4 \end{vmatrix} = 0}_{M_{34,34}} \quad \geq 0 \quad \checkmark \text{ ok}$$

$$\Delta_3 = \underbrace{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 9 \end{vmatrix} = 0}_{D_3 = M_{123,123}},\ \underbrace{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & -2 \\ 0 & -2 & 4 \end{vmatrix} = 0}_{M_{124,124}},\ \underbrace{\begin{vmatrix} 1 & 0 & 0 \\ 0 & 9 & -6 \\ 0 & -6 & 4 \end{vmatrix} = 0}_{M_{134,134}},\ \underbrace{\begin{vmatrix} 1 & 3 & -2 \\ 3 & 9 & -6 \\ -2 & -6 & 4 \end{vmatrix} = 0}_{M_{234,234}} \geq 0 \quad \text{ok}\checkmark$$

$$\Delta_4 = \underbrace{\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 3 & -2 \\ 0 & 3 & 9 & -6 \\ 0 & -2 & -6 & 4 \end{vmatrix}}_{D_4 = |A|} = 0 \geq 0 \quad \text{ok}\checkmark$$

**Conclusion:**
$\Delta_i \geq 0$ for all $i$
$f$ is pos. semidef.

(1)

**Reduced rank criterion:** (2016)

If $A$ is an $n \times n$-matrix (symmetric) with $\underbrace{\operatorname{rk} A}_{=r} < n$

then:

$D_1, D_2, \dots, D_r > 0 \implies A$ pos. semidefinite

$D_1 < 0, D_2 > 0, \dots$

$\dots, (-1)^r \cdot D_r > 0 \implies A$ neg. semidefinite

**Ex:**

$$A = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 4 \end{pmatrix}$$

$D_1 = 3$

$D_2 = 3$

$D_3 = 0$

$A$ is pos. semidefinite by the RRC

$\operatorname{rk} A = 2$

**Ex:**

$$A = \begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 0 & 1 \end{pmatrix}$$

$$\downarrow$$

$$\begin{pmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\operatorname{rk} A = 2$

$\left.\begin{matrix} D_1 = 1 \\ D_2 = 1 \end{matrix}\right\} > 0$

$D_3 = 0$

$D_4 = 0$

RRC: $A$ pos. semidefinite.

## 2 Unconstrained optimization

max/min $f(x_1, \ldots, x_n)$ ← max/min = global max/min

**Method:**

For 'nice' functions $f$:
$\underline{x}^*$ max/min for $f$
⇓
$\underline{x}^*$ stationary pt. for $f$

1. Find all stationary pts for $f$:

$$\underbrace{f'_{x_1} = f'_{x_2} = \ldots = f'_{x_n} = 0}$$

FOC = first order conditions.

Stationary pts = candidates for max/min

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Ex: $f(x,y) = x^2y^3 + y^2 - 2y$

$f'_x = 2xy^3 = 0 \Rightarrow x \cdot y^3 = 0$ $\underline{x=0}$ or $\underline{y=0}$

$f'_y = x^2 \cdot 3y^2 + 2y - 2 = 0$

$\underline{x=0}$: $2y - 2 = 0$, $y = 1$ $\Big\} \Rightarrow (x,y) = (0,1)$

$\underline{y=0}$: $-2 = 0$ impossible

Stationary pts:
$(x,y) = \underline{\underline{(0,1)}}$

## ② Local classification of stationary pts:

**Second derivative test:**

If $\underline{x}^* = (x_1^*, x_2^*, \dots, x_n^*)$ is a stationary pt of $f$, then:

$H(f)(\underline{x}^*)$ pos defn. $\Rightarrow$ $\underline{x}^*$ local min.
$H(f)(\underline{x}^*)$ neg. defn. $\Rightarrow$ $\underline{x}^*$ local max.
$H(f)(\underline{x}^*)$ indefinite $\Rightarrow$ $\underline{x}^*$ saddle point

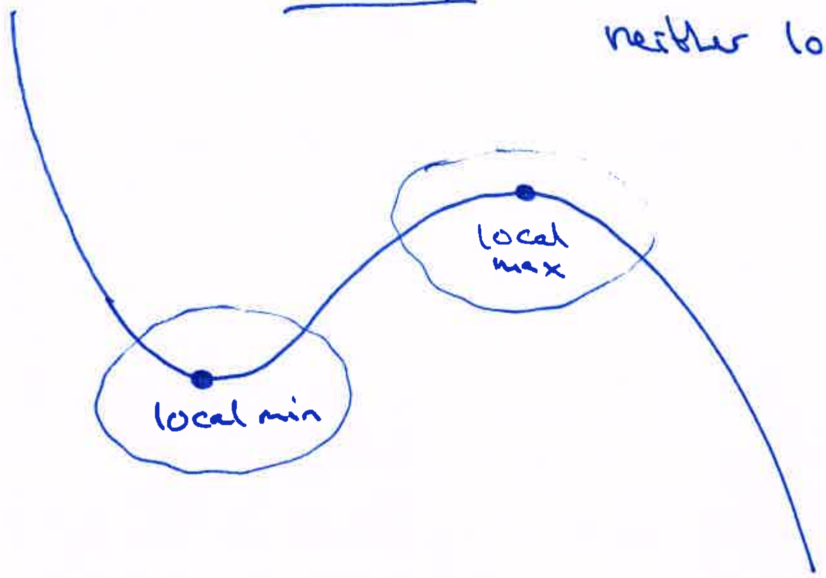
Otherwise, the test is inconclusive.

**Defn.:**

$\underline{x}^*$ is local min if $f(\underline{x}^*) \leq f(\underline{x})$ for all $\underline{x}$ close to $\underline{x}^*$

—"— local max $f(\underline{x}^*) \geq f(\underline{x})$ —"— ———

$\underline{x}^*$ is saddle pt if it is a stationary pt that is neither local max nor min



Ex: $f(x,y) = x^2y^3 + y^2 - 2y$

$f'_x = 2xy^3$

$f'_y = 3x^2y^2 + 2y - 2$

$$H(f) = \begin{pmatrix} f''_{xx} & f''_{xy} \\ f''_{yx} & f''_{yy} \end{pmatrix} = \begin{pmatrix} 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y+2 \end{pmatrix}$$

for any "nice" fn. in $n$ variables, $H(f)$ is a symmetric $n \times n$ - matrix

$C^2$

Stat. pts.: $(x,y) = (0,1)$

$$H(f)(0,1) = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$D_1 = 2$
$D_2 = 4$

pos. def.
$\Downarrow$
$(x,y) = (0,1)$ is local min

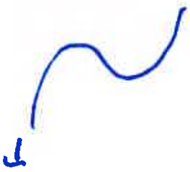
③ Determine if local max/min are also global max/min

— $f(x,y) = x^2y^3 + y^2 - 2y$ has no global max

— Is $(0,1)$ global min: $f(0,1) = 0 + 1 - 2 = -1$

$x = 1$: $f(1,y) = y^3 + y^2 - 2y$

$y = -10$:
$f(1,-10) = -1000 + 100 + 20$
$= -880$



$-\infty$

Concl: no global min either

Note: (1) global max $\Rightarrow$ local max
global min $\Rightarrow$ local min
} local classification gives partial results on global max/min.

(2) Second derivative test + FOC:

local data on $f$ since these data are local $\longrightarrow$ $\boxed{f'_{x_1}(\underline{x}^*) = \cdots = f'_{x_n}(\underline{x}^*) = 0 \text{ and } H(f)(\underline{x}^*) \begin{cases} \text{pos. def.} \\ \text{neg. def.} \\ \text{indefinite} \end{cases}}$

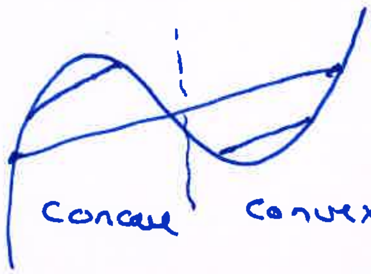
local data
$\Downarrow$
local classification

## 3 Convex and concave functions

Ex: $f(x)$ fn. in one variable


$f$ convex: $f''(x) \geq 0$ for all $x$

$f$ concave $f''(x) \leq 0$ for all $x$




globally neither convex nor concave.

Defn: $f(x_1, \ldots, x_n)$ fn. in $n$ variables

$f$ is convex if: For any two pts $P, Q$ on the graph of $f$, the line segment $[P, Q]$ lies over (or on) the graph of $f$

$f$ is concave if: For any two pts. $P, Q$ on the graph of $f$, the line segment $[P, Q]$ lies under (or on) the graph of $f$.

Result:

$f$ is convex $\iff$ $H(f)(\underline{x})$ is positive semi-defn. for all pts $\underline{x}$.

$f$ is concave $\iff$ $H(f)(\underline{x})$ is negative semi-defn. for all pts. $\underline{x}$.

Ex: $f(x,y) = x^2y^3 + y^2 - 2y$

$$H(f) = \begin{pmatrix} 2y^3 & 6xy^2 \\ 6xy^2 & 6x^2y + 2 \end{pmatrix}$$

$D_1 = 2y^3$

$f$ convex: $2y^3 \geq 0$ for all $(x,y)$ no

$f$ concave: $2y^3 \leq 0$ for all $(x,y)$

Result:

$\cup$ $f$ convex $\Rightarrow$ any stationary pt is global min

$\cap$ $f$ concave $\Rightarrow$ any stationary pt. is global max

**Ex:** $f(x,y,z) = x^4 + 2x^2 + y^2 + 2z^2 + z^4 - 4xz$

$\boxed{\text{max/min } f(x,y,z)}$

① **Find stationary pts.:**

FOC
$$\begin{cases} f'_x = 4x^3 + 4x - 4z = 0 \\ f'_y = 2y = 0 \\ f'_z = 4z + 4z^3 - 4x = 0 \end{cases}$$

$x^3 + x = z$

$y = 0$

$z + z^3 = x$

$$\boxed{\begin{aligned} x^3 + x &= z \\ z + z^3 &= x \end{aligned}}$$

$$\Downarrow$$

$$\cancel{x^3 + x + (x^3 + x)^3 = x}$$

$(x^3+x) - (z+z^3) = z - x$

$(x^3 - z^3) + (x - z) = (z - x)$

$(x^3 - z^3) + \underbrace{(x-z) + (x-z)}$ $= 0$

$(x-z)\cdot(x^2 + xz + z^2) + 2(x-z)$

$(x^3 - z^3) : (x - z) = x^2 + xz + z^2$
$-(x^3 - x^2 z)$
$x^2 z - z^3$
$-(x^2 z - x z^2)$
$x z^2 - z^3$
$x z^2 - z^3$
$0$

**Stationary pts:**

$(x,y,z) = \underline{(0,0,0)}$

$(x^2 + xz + z^2)(x - z) + 2(x - z) = 0$

$(x - z)\cdot(x^2 + xz + z^2 + 2) = 0$

$x = z$ or $x^2 + xz + z^2 = -2$

$x^2 + xz + z^2 \geq 0$ for all $x, z$

$x^3 + x = x$
$x^3 = 0$
$x = 0, z = 0$

$(x\ z)\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1 \end{pmatrix}\begin{pmatrix} x \\ z \end{pmatrix} = -2$

$\Delta_1 = 1$
$\Delta_2 = 1 - 1/4 = 3/4 > 0$

no solutions

since $x^2 + xz + z^2$ is pos. def.

(2) $f'_x = 4x^3 + 4x - 4z$

$f'_y = 2y$

$f'_z = 4z + 4z^3 - 4x$

$$H(f) = \begin{pmatrix} f''_{xx} & f''_{xy} & f''_{xz} \\ f''_{xy} & f''_{yy} & f''_{yz} \\ f''_{xz} & f''_{yz} & f''_{zz} \end{pmatrix} = \begin{pmatrix} 12x^2+4 & 0 & -4 \\ 0 & 2 & 0 \\ -4 & 0 & 12z^2+4 \end{pmatrix}$$

$$H(f)(0,0,0) = \begin{pmatrix} 4 & 0 & -4 \\ 0 & 2 & 0 \\ -4 & 0 & 4 \end{pmatrix}$$

$D_1 = 4$
$D_2 = 8$
$D_3 = 2 \cdot 0 = 0$

RREC: rk = 2, pos. semi defn.

Second derivative test: inconclusive

must check this condition:

Is f convex: $\boxed{H(f)(x,y,z) \text{ pos semidefn for all } x,y,z}$

$$\begin{pmatrix} 12x^2+4 & 0 & -4 \\ 0 & 2 & 0 \\ -4 & 0 & 12z^2+4 \end{pmatrix}$$

$D_1 = \underbrace{12x^2}_{\geq 0} + 4 > 0$ for all $x,y,z$

$D_2 = 2(12x^2+4) > 0$ — " —

$D_3 = 2 \cdot \left[ (12x^2+4)(12z^2+4) - 16 \right]$

$= 2\left(144x^2z^2 + 48x^2 + 48z^2 + \cancel{16} - \cancel{16}\right)$

$= 288x^2z^2 + 96x^2 + 96z^2 \geq 0$ for all $x,y,z$

$(x,z) = (0,0)$: $D_3 = 0$

$D_1, D_2 > 0$, $D_3 = 0$

RREC: positive semidefn.

f convex

$(x,z) \neq (0,0)$: $D_3 > 0$

pos. defn. for all $x,y,z$

f convex

$\Rightarrow$ f is convex

$f(0,0,0) = 0$ is global min