Plan:

1. Constrained optimization
2. Lagrange problems
3. Kuhn-Tucker problems

Monday (17-20): Plenary Session II

Next Friday: Midterm Exam 1500-1600

Important: definiteness ; $\boxed{\text{Lecture 1-6, Problem 1-6}}$

Review:

a) Unconstrained optimization: max/min $f(\underline{x}) = f(x_1, \ldots, x_n)$

Method:

① Find stationary pts:

$$\text{FOC} \begin{cases} f'_{x_1} = 0 \\ \vdots \\ f'_{x_n} = 0 \end{cases}$$

Solutions = stationary pts (candidate pts)

Tip: Compute $f(\underline{x}^*)$ for each cand. pt. $\underline{x}^*$

If f is "nice":

$\underline{x}^*$ is max/min
$\Downarrow$
$\underline{x}^*$ stationary pt

② Local classification:

$$H(f) = \begin{pmatrix} f''_{x_1x_1} & f''_{x_1x_2} & \cdots & f''_{x_1x_n} \\ \vdots & & & \vdots \\ f''_{x_nx_1} & f''_{x_nx_2} & \cdots & f''_{x_nx_n} \end{pmatrix}$$

$n \times n$ symmetric matrix

Second derivative test:

If $\underline{x}^*$ is a stationary pt, then:

$H(f)(\underline{x}^*)$ pos. defn. $\Rightarrow$ $\underline{x}^*$ local min

$H(f)(\underline{x}^*)$ neg. defn. $\Rightarrow$ $\underline{x}^*$ local max

$H(f)(\underline{x}^*)$ indefinite $\Rightarrow$ $\underline{x}^*$ saddle pt

Otherwise, the test is inconclusive. Use defn.

③ Determine global max/min

systematic method: check if f is convex/concave

ad hoc methods: use defn.

b) Convex / concave functions

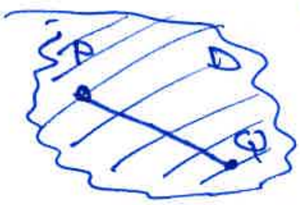
f convex $\iff$ $H(f)(\underline{x})$ is positive semidef. for all $\underline{x}$
f concave $\iff$ $H(f)(\underline{x})$ is negative — " — —

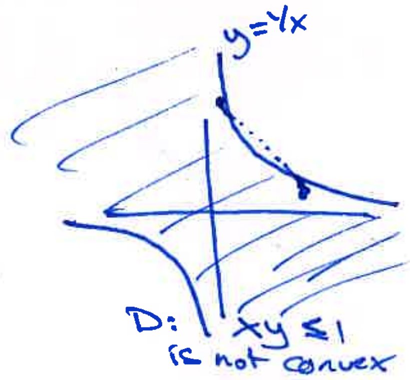
Convex / concave optimization:

f convex $\implies$ any stationary pt is a global min
f concave $\implies$ any stationary pt is a global max

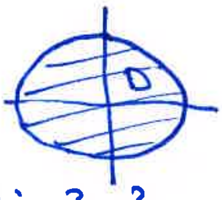
Convex sets:

D is a convex set if, for any pts $P, Q \in D$, the line segment $[P,Q]$ is included in D





D: $xy \leq 1$ is not convex



D: $x^2+y^2 \leq 1$ is convex

D: $x^2+y^2 \geq 1$ is not convex

f is a convex function
$\Updownarrow$
the set of all pts lying on or over the graph of f is a convex set

f is a concave function $\iff$ the set of all pts lying on or under the graph of f is a convex set.

f convex

f not convex

Note:

A is positive defn. $\Rightarrow$ A is pos. semidefn.

$\Updownarrow$ $\qquad\qquad\qquad\qquad$ $\Updownarrow$

$\underline{x}^T A \underline{x} > 0 \quad (\underline{x} \neq 0)$ $\qquad\qquad$ $\underline{x}^T A \underline{x} \geq 0 \quad (\text{all } \underline{x})$

A is negative defn. $\Rightarrow$ A is neg. semidefn.

① Constrained optimization:

Ex: i) max $f(x,y) = xy$ when $x^2+y^2 = 2$

ii) min $f(x,y) = x^2+y^2$ when $xy = 1$

$f(\underline{x}) = f(x_1,\dots,x_n)$: objective function

$\left.\begin{array}{l} g(\underline{x}) = g(x_1,\dots,x_n) = a \\ g(\underline{x}) = g(x_1,\dots,x_n) \leq a \end{array}\right\}$: constraints

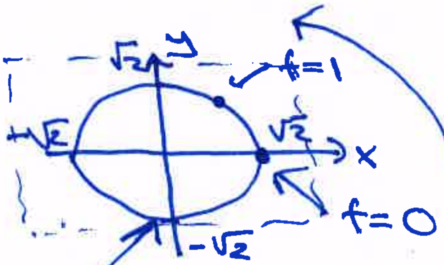
max $f(x,y) = xy$ on D

Ex i):

$x^2+y^2 = 2$

circle, $r = \sqrt{2}$

center $(0,0)$



$D = \{(x,y) : x^2+y^2 = 2\}$

set of admissible pts

D is compact.

$z = f(x,y)$

$z = xy$

Defn: A pt. is called admissible if all constraints are satisfied

Extreme Value Theorem (EVT)

If f is a continuous function on a compact set D, then f has a (global) max/min.

Defn:

A set D in $\mathbb{R}^n$ is compact if it is closed and bounded.

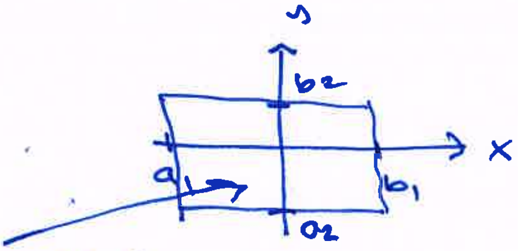
It is closed if it is defined by equations (=) or closed inequalities ($\leq, \geq$)

It is bounded if there are constants $a_1, a_2, \ldots, a_n$ (bounded) and $b_1, b_2, \ldots, b_n$ such that

$$a_1 \leq x_1 \leq b_1$$
$$a_2 \leq x_2 \leq b_2$$
$$\vdots$$
$$a_n \leq x_n \leq b_n$$

for all pts $(x_1, \ldots, x_n)$ in D.

n=2: $a_1 \leq x \leq b_1$
$a_2 \leq y \leq b_2$



the set D is contained in a finite rectangle

Ex: max $f(x,y) = xy$ when $x^2+y^2=2$

$\mathcal{L} = xy - \lambda\cdot(x^2+y^2)$ Lagrangian $\left\{ \begin{array}{l} x,y: \text{ proper variables} \\ \lambda: \text{ Lagrange multiplier} \end{array} \right.$

$$\begin{array}{ll} \mathcal{L}'_x = & y - \lambda\cdot 2x = 0 \\ \mathcal{L}'_y = & x - \lambda\cdot 2y = 0 \\ & x^2+y^2 = 2 \end{array}$$

$\left.\right\}$ FOC, $\}$ C

FOC + C = Lagrange conditions

(1) $y = 2\lambda\cdot x$

(2) $x - 2\lambda\cdot y = 0$

$x - 2\lambda\cdot(2\lambda x) = 0$

$x - 4\lambda^2\cdot x = 0$

$x(1-4\lambda^2) = 0$

$x = 0$ or $4\lambda^2 = 1$

$\lambda^2 = 1/4$

$x=0$ or ($\lambda = 1/2$ or $\lambda = -1/2$)

a) $x = 0$: ~~$y = 0$, $0^2+0^2 = 2$ impossible~~

b) $\lambda = 1/2$: $y = x$

$2x^2 = 2$

$x^2 = 1$

$x = \pm 1$

$(x,y;\lambda) = (1,1;1/2), (-1,-1;1/2)$

c) $\lambda = -1/2$: $y = -x$

$2x^2 = 2$

$x = \pm 1$

$(x,y;\lambda) = (1,-1;-1/2), (-1,1;-1/2)$

Candidate pts:

$(1,1;1/2)$, $(-1,-1;1/2)$, ~~$(1,-1;-1/2)$, $(-1,1;-1/2)$~~

$f=1$, $f=1$, ~~$f=-1$~~, ~~$f=-1$~~

Conclusion: $f_{max} = 1$ at $(x,y) = (1,1), (-1,-1)$

(since D is a circle and compact)

② Lagrange problems (= equality constraints)

General form:

$$\boxed{\max/\min\ f(x_1,\dots,x_n) \quad \text{when} \quad \begin{cases} g_1(x_1,\dots,x_n)=a_1 \\ g_2(x_1,\dots,x_n)=a_2 \\ \vdots \\ g_m(x_1,\dots,x_n)=a_m \end{cases}}$$

Method: $\mathcal{L} = f(\underline{x}) - \lambda_1 \cdot g_1(\underline{x}) - \lambda_2 \cdot g_2(\underline{x}) - \dots - \lambda_m \cdot g_m(\underline{x})$

(variables: $\underbrace{x_1, x_2, \dots, x_n}_{\text{proper var.}}$; $\underbrace{\lambda_1, \lambda_2, \dots, \lambda_m}_{\text{Lagrange mult.}}$)

① Find candidate pts:

$$\text{FOC:} \begin{cases} \mathcal{L}'_{x_1} = 0 \\ \mathcal{L}'_{x_2} = 0 \\ \vdots \\ \mathcal{L}'_{x_n} = 0 \end{cases} \qquad \text{C:} \begin{cases} g_1(\underline{x}) = a_1 \\ g_2(\underline{x}) = a_2 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$$

Lagrange conditions = FOC + C

m + n equations in m + n variables

Solutions of FOC + C = candidate pts

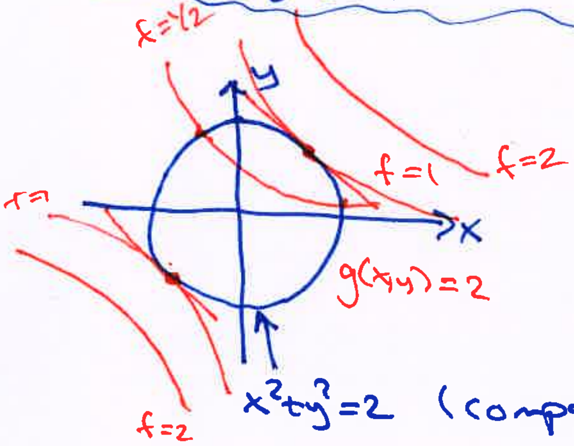
② Check if D is compact: $D = \{(x_1,\dots,x_n) : g_1(\underline{x})=a_1, \ \dots, \ g_m(\underline{x})=a_m \}$

(that is, is D bounded) — admissible points

$$\boxed{\begin{array}{l} \textbf{Thm: } \text{Necessary conditions for Lagrange problems} \\ \text{If } \underline{x}^* \text{ is a max/min in a Lagrange problem,} \\ \text{then we have:} \\ \text{i) } (\underline{x}^*; \underline{\lambda}^*) \text{ satisfies FOC+C for some } \underline{\lambda}^*, \ \underline{\underline{\text{or}}} \\ \text{ii) NDCQ fails at } \underline{x}^* \end{array}}$$

NDCQ: will talk about this next week

$\underline{x}^*$ max/min in Lagrange problem (NDCQ holds) $\Longrightarrow$ $(\underline{x}^*, \underline{\lambda}^*)$ satisfies FOC+C



$x^2+y^2=2$ (compact)

max $f(x,y)=xy$ when $x^2+y^2=2$

Cand. pts:

$(1,1;1/2)$, $(-1,-1;1/2)$ $f=1$

$(1,-1;-1/2)$, $(-1,1;-1/2)$ $f=-1$

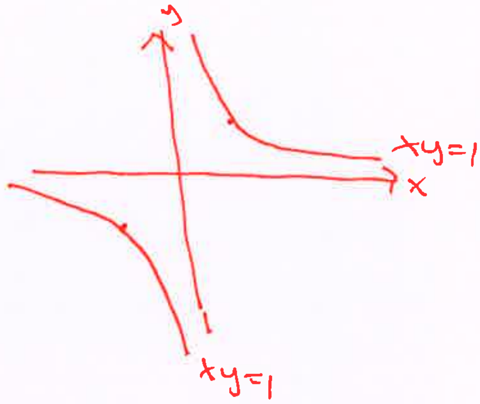
$(x,y)=(1,1)$: $f=1$ $\rightarrow$ Level curve: $f(x,y)=1$

$=(-1,-1)$

$xy=1$

$y=1/x$

Ex: ii) min $f(x,y) = x^2 + y^2$ when $xy = 1$



Candidate pts:

$h = x^2 + y^2 - \lambda xy$

$$\boxed{\begin{aligned} h'_x &= 2x - \lambda y = 0 \\ h'_y &= 2y - \lambda x = 0 \\ xy &= 1 \end{aligned}}$$

} FOC

} C

$D = \{(x,y) : xy = 1\}$

$\Updownarrow$

$y = 1/x$

hyperbola

D is <u>not</u> bounded

(1) $2x = \lambda y$

$x = \frac{\lambda}{2} \cdot y$

(2) $2y - \lambda x = 0$

$2y - \lambda\left(\frac{\lambda}{2} y\right) = 0 \quad | \cdot 2$

$4y - \lambda^2 y = 0$

$(4 - \lambda^2) y = 0$

$y(2-\lambda)(2+\lambda) = 0$

$y = 0$ or	$\lambda = 2$ or	$\lambda = -2$
$x = 0$	$x = y$	$x = -y$
$xy = 0 \cdot 0 \neq 1$	$x^2 = 1 \quad x = \pm 1$	$(-y)y = 1$
not possible	$(x,y;\lambda) = (1,1;2)$,	$y^2 = -1$
no sol'n	$(-1,-1;2)$	not possible
		no sol'n.

Candidate pts:

$(x,y;\lambda) = \underline{(1,1;2)}$, $\underline{(-1,-1;2)}$

$f = 2$ $\quad$ $f = 2$

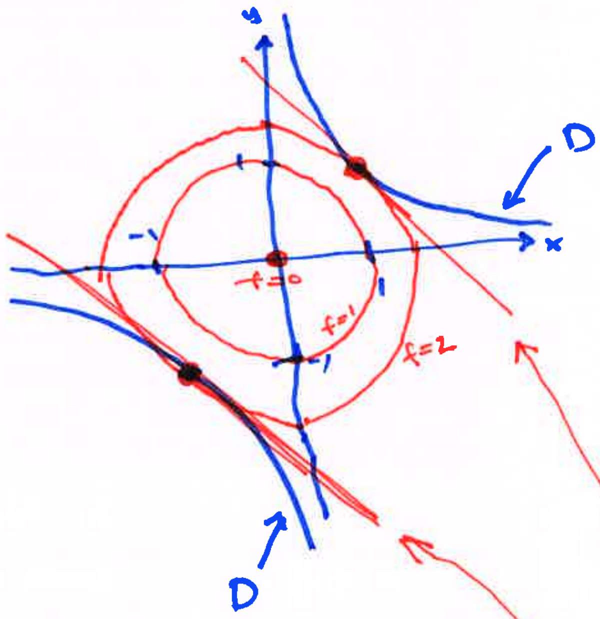
D <u>not</u> compact.

How can we check if $f_{max} = 2$?

Explanation of Lagrange conditions $L'_x = L'_y = 0$

Lagrange problem: min $f = x^2 + y^2$ when $xy = 1$

① Adm. pts: $\underset{g(x,y)}{xy} = \underset{a}{1}$



② Values of $f(x,y) = x^2 + y^2$ (objective fn).

Level curves: $f(x,y) = c$

c=0: $f(x,y) = 0$: $x^2 + y^2 = 0$
$(x,y) = (0,0)$

c=1: $f(x,y) = 1$ $x^2 + y^2 = 1$
circle, center $(0,0)$
rad. 1

no adm. pts with $f = 0$ or $f = 1$

c=2: $f(x,y) = 2$ $x^2 + y^2 = 2$
circle, center $(0,0)$
radius $\sqrt{2}$ — two adm. pts. $(1,1), (-1,-1)$ with $f = 2$

Minimum: D: $xy = 1$ and level curve $f(x,y) = c$ meet in a point where the tangent line of $g(x,y) = a$ and of $f(x,y) = c$ are the same

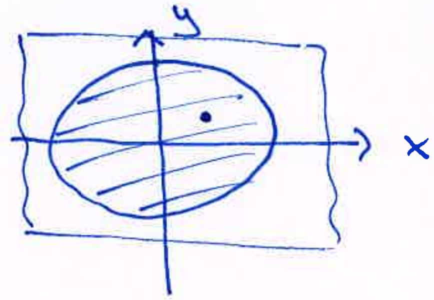
tangent lines have the same slope $\Rightarrow -\dfrac{f'_x}{f'_y} = -\dfrac{g'_x}{g'_y} \Leftrightarrow \begin{cases} f'_x = \lambda \cdot g'_x \\ f'_y = \lambda \cdot g'_y \end{cases} \Leftrightarrow \begin{cases} L'_x = 0 \\ L'_y = 0 \end{cases}$

(3) Kuhn-Tucker problems

(= closed inequality constraints $\leq$ $\geq$)

Ex:

$\max f(x,y) = x + 3y$ when $x^2 + y^2 \leq 10$



$x^2+y^2=10$: circle
$x^2+y^2<10$: inside
compact

Candidate pts:

$$L = x + 3y - \lambda \cdot (x^2+y^2)$$

FOC:
$$\begin{cases} L'_x = 1 - \lambda \cdot 2x = 0 \\ L'_y = 3 - \lambda \cdot 2y = 0 \end{cases}$$

C: $x^2 + y^2 \leq 10$

CSC:
$$\left\{ \begin{array}{c} \lambda \geq 0 \\ \text{and} \\ \lambda (x^2+y^2-10) = 0 \end{array} \right\}$$

$\Updownarrow$

$x^2+y^2=10$: $\lambda \geq 0$

$x^2+y^2<10$: $\lambda = 0$

CSC = complementary slackness condition

standard form of KT problem

= max (not min) + $\leq$ (not $\geq$)

Kuhn-Tucker conditions: FOC + C + CSC

Ex:

$$\left.\begin{aligned} L'_x &= 1 - 2\lambda x = 0 \\ L'_y &= 3 - 2\lambda y = 0 \end{aligned}\right\} \text{FOC}$$

$$x^2 + y^2 \leq 10 \quad \text{C}$$

$$\lambda \geq 0, \ \lambda(x^2+y^2-10) = 0 \quad \text{CSC}$$

$$\Uparrow$$

	a)	b)
C:	$x^2+y^2=10$	$x^2+y^2<10$
CSC:	$\lambda \geq 0$	$\lambda = 0$
FOC:	$1-2\lambda x = 0$ $3 - 2\lambda y = 0$	$1-2\lambda x=0$ $3-2\lambda y = 0$

(1) $x = \frac{1}{2\lambda}$ $(\lambda \neq 0)$

(2) $y = \frac{3}{2\lambda}$

$$\left(\frac{1}{2\lambda}\right)^2 + \left(\frac{3}{2\lambda}\right)^2 = 10$$

$$\frac{1^2+3^2}{(2\lambda)^2} = 10$$

$$\frac{10}{(2\lambda)^2} = 10$$

$$(2\lambda)^2 = 1$$

$$2\lambda = \pm 1$$

$$\lambda = \pm 1/2$$

$\lambda = 0$: $1 = 0$ impossible

a) $(x, y; \lambda) =$

$(1, 3; 1/2)$,

$\cancel{(-1, -3; -1/2)}$ $\leftarrow \lambda < 0$

All cand. pts:

$(x, y; \lambda) = (1, 3; 1/2)$

$f = 10$

Concl:

$f_{max} = 10$

at $(x, y) = (1, 3)$

since D is compact.

Kuhn-Tucker formulation:

A Kuhn-Tucker problem in standard form can be written:

$$\max f(\underline{x}) = f(x_1, \ldots, x_n) \quad \text{when} \quad \begin{cases} g_1(\underline{x}) \le a_1 \\ g_2(\underline{x}) \le a_2 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{cases}$$

Std form: $\begin{cases} \max & (\text{not min}) \\ \le & (\text{not } \ge) \end{cases}$

Method: $\mathcal{L} = f(\underline{x}) - \lambda_1 \cdot g_1(\underline{x}) - \lambda_2 g_2(\underline{x}) - \ldots - \lambda_m g_m(\underline{x})$

(same as Lagrange problem)

FOC: (same as Lagrange case) $\begin{cases} \mathcal{L}'_{x_1} = 0 \\ \mathcal{L}'_{x_2} = 0 \\ \vdots \\ \mathcal{L}'_{x_n} = 0 \end{cases}$

C: $\begin{cases} g_1(\underline{x}) \le a_1 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{cases}$

CSC: $\begin{cases} \lambda_1 \ge 0 \text{ and } \lambda_1(g_1(\underline{x}) - a_1) = 0 \\ \lambda_2 \ge 0 \text{ and } \lambda_2(g_2(\underline{x}) - a_2) = 0 \\ \vdots \\ \lambda_m \ge 0 \text{ and } \lambda_m(g_m(\underline{x}) - a_m) = 0 \end{cases}$

(when we use a std. form)

Kuhn-tucker conditions: $\boxed{\text{FOC} + \text{C} + \text{CSC}}$

Solutions of FOC + C + CSC = Candidate pts.

Result:

$\underline{x}^*$ is max $\underset{\text{NDCQ holds}}{\Longrightarrow}$ $(\underline{x}^*; \underline{\lambda}^*)$ satisfies FOC + C + CSC for some $\underline{\lambda}^*$.

Ex: KT problem not in std form

$$\min\ x^2+y^2 \quad \text{when} \quad xy \geq 1$$

Transform it to a std form:

a) max: $\min\ x^2+y^2 = \max\ -(x^2+y^2)$

b) $\leq$: $xy \geq 1 \quad | \cdot (-1)$

$$-xy \leq -1$$

$$\boxed{\text{Std. form:} \quad \max\ -f(x,y) = -(x^2+y^2) \quad \text{when} \quad -xy \leq -1}$$