Plan:

1. Second order conditions (SOC)
2. NDCQ: non-degenerate constraint qualification

Review:

a) Lagrange problems

max/min $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) = a_1 \\ g_2(\underline{x}) = a_2 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$

" $f(x_1, \dots, x_n)$

$\mathcal{L} = f(\underline{x}) - \lambda_1 g_1(\underline{x}) - \dots - \lambda_m \cdot g_m(\underline{x})$

FOC: $\begin{cases} \mathcal{L}'_{x_1} = 0 \\ \vdots \\ \mathcal{L}'_{x_n} = 0 \end{cases}$ C: $\begin{cases} g_1(\underline{x}) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases}$

Solution of FOC + C (= Lagrange cond.) give us candidate points.

$\Downarrow$

Extreme Value Theorem (EVT)

If f is continuous on a compact (closed and bounded) set D, then f attains a maximum and a minimum.

b) Kuhn-Tucker problems

std. form: max $f(\underline{x})$ when $\begin{cases} g_1(\underline{x}) \le a_1 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{cases}$

$\mathcal{L} = f(\underline{x}) - \lambda_1 g_1(\underline{x}) - \dots - \lambda_m g_m(\underline{x})$

FOC: $\begin{cases} \mathcal{L}'_{x_1} = 0 \\ \vdots \\ \mathcal{L}'_{x_n} = 0 \end{cases}$ C: $\begin{cases} g_1(\underline{x}) \le a_1 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{cases}$

CSC: $\begin{cases} \lambda_1 \ge 0 \text{ and } \lambda_1 \cdot (g_1(\underline{x}) - a_1) = 0 \\ \vdots \\ \lambda_m \ge 0 \text{ and } \lambda_m \cdot (g_m(\underline{x}) - a_m) = 0 \end{cases}$

Solutions of FOC + C + CSC (= KT conditions) give us candidate pts

$\Leftarrow$

Thm:
If $\underline{x}^*$ is max/min in Lagrange / Kuhn-Tucker problem **and** NDCQ is satisfied at $\underline{x}^*$, then $(\underline{x}^*, \underline{\lambda}^*)$ is a candidate pt (one of the pts satisfying FOC+C / FOC+C+CSC)

Ex: $\min f(x,y,z) = 2x^2+y^2+3z^2$ when $\begin{cases} x-y+2z \geq 3 & |\cdot(-1) \\ x+y \geq 3 & |\cdot(-1) \end{cases}$

std. form →
$$= \boxed{\max -f(x,y,z) = -2x^2-y^2-3z^2 \text{ when } \begin{cases} -x+y-2z \leq -3 \\ -x-y \leq -3 \end{cases}}$$

$$L = -2x^2-y^2-3z^2 - \lambda_1\cdot(-x+y-2z) - \lambda_2(-x-y)$$
$$= -2x^2-y^2-3z^2 + \lambda_1(x-y+2z) + \lambda_2(x+y)$$

FOC	C	CSC
$L'_x = -4x+\lambda_1+\lambda_2 = 0$	$x-y+2z \geq 3$	$\lambda_1 \geq 0,\ \lambda_1\cdot(x-y+2z-3)=0$
$L'_y = -2y-\lambda_1+\lambda_2 = 0$	$x+y \geq 3$	$\lambda_2 \geq 0,\ \lambda_2\cdot(x+y-3)=0$
$L'_z = -6z+2\lambda_1 = 0$		

a) $x-y+2z=3$, $x+y=3$	b) $x-y+2z=3$, $x+y>3$	c) $x-y+2z>3$, $x+y=3$	d) $x-y+2z>3$, $x+y>3$
$\lambda_1 \geq 0$, $\lambda_2 \geq 0$	$\lambda_1 \geq 0$, $\lambda_2 = 0$	$\lambda_1 = 0$, $\lambda_2 \geq 0$	$\lambda_1 = 0$, $\lambda_2 = 0$
$-4x+\lambda_1+\lambda_2=0$, $-2y-\lambda_1+\lambda_2=0$, $-6z+2\lambda_1=0$	← "	← "	← "
			⇓ $-4x=0$, $-2y=0$, $-6z=0$; ~~$x=0$, $y=0$, $z=0$, $\lambda_1=0$, $\lambda_2=0$~~
			$0>3$ not possible

a)

$$\begin{aligned} x - y + 2z &= 3 \\ x + y &= 3 \\ -4x + \lambda_1 + \lambda_2 &= 0 \\ -2y - \lambda_1 + \lambda_2 &= 0 \\ -6z + 2\lambda_1 &= 0 \end{aligned}$$

$$\boxed{\lambda_1 \geq 0,\ \lambda_2 \geq 0}$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 3 \\ 1 & 1 & 0 & 0 & 0 & 3 \\ -4 & 0 & 0 & 1 & 1 & 0 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & -6 & 2 & 0 & 0 \end{array}\right) \begin{array}{l} -1 \\ 4 \end{array}$$

$$\left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 3 \\ 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & -4 & 8 & 1 & 1 & 12 \\ 0 & -2 & 0 & -1 & 1 & 0 \\ 0 & 0 & -6 & 2 & 0 & 0 \end{array}\right) \begin{array}{l} 2 \\ 1 \end{array} \rightarrow \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 3 \\ 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & 4 & 1 & 1 & 12 \\ 0 & 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & -6 & 2 & 0 & 0 \end{array}\right)$$

$$\rightarrow \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 3 \\ 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & 4 & 1 & 1 & 12 \\ 0 & 0 & -6 & 2 & 0 & 0 \end{array}\right) \begin{array}{l} 2 \\ -3 \end{array} \rightarrow \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 3 \\ 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 3 & 12 \\ 0 & 0 & 0 & 5 & -3 & 0 \end{array}\right) 5$$

$$\rightarrow \begin{array}{c} \left(\begin{array}{ccccc|c} 1 & -1 & 2 & 0 & 0 & 3 \\ 0 & 2 & -2 & 0 & 0 & 0 \\ 0 & 0 & -2 & -1 & 1 & 0 \\ 0 & 0 & 0 & -1 & 3 & 12 \\ 0 & 0 & 0 & 0 & 12 & 60 \end{array}\right) \\ \begin{array}{ccccc} x & y & z & \lambda_1 & \lambda_2 \end{array} \end{array}$$

$x = \underline{2}$

$2y = 2z = 2 \Rightarrow y = \underline{1}$

$-2z = -2 \Rightarrow \underline{z = 1}$

$-\lambda_1 = -3 \Rightarrow \lambda_1 = \underline{3}$

$\lambda_2 = \frac{60}{12} = \underline{5}$

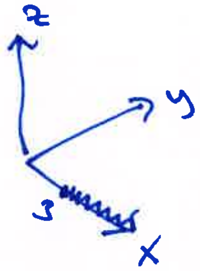
$\Big\} \geq 0$

a) $(x, y, z; \lambda_1, \lambda_2) = \underline{(2, 1, 1; 3, 5)}$

$\boxed{-f = -12}$

Wait with case b), c):

Is D bounded: D: $\begin{cases} x-y+z \geq 3 \\ x+y \geq 3 \end{cases}$



$y=z=0$: $x \geq 3$

⇕

$(x,0,0)$ with $x \geq 3$ are in D $\Rightarrow$ not bounded

① Second order condition

SOC: If $(\underline{x}^*; \underline{\lambda}^*)$ satisfies FOC+C / FOC+C+CSC, then: Consider the function

$$h(\underline{x}) = \mathcal{L}(x_1, x_2, \ldots, x_n; \underbrace{\lambda_1^*, \lambda_2^*, \ldots, \lambda_m^*}_{\text{values of } \lambda\text{'s from the candidate pt.}})$$

$h(\underline{x})$ concave fn. $\Rightarrow$ $\underline{x}^*$ is max

$h(\underline{x})$ convex fn. $\Rightarrow$ $\underline{x}^*$ is min

Ex: $(\underbrace{2,1,1}_{\underline{x}^*};\underbrace{3,5}_{\underline{\lambda}^*})$ candidate pt.

$\mathcal{L} = -2x^2 - y^2 - 3z^2 + \lambda_1(x-y+z) + \lambda_2(x+y)$

$h = -2x^2 - y^2 - 3z^2 + 3(x-y+z) + 5(x+y)$

concave? Yes.

SOC: $(-f)_{max} = -12$ at $(x,y,z) = (2,1,1)$

$f_{min} = 12$ at $(x,y,z) = (2,1,1)$

$$\left.\begin{aligned} h'_x &= -4x + 3 + 5 \\ h'_y &= -2y - 3 + 5 \\ h'_z &= -6z + 6 \end{aligned}\right\}$$

$$H(h) = \begin{pmatrix} -4 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -6 \end{pmatrix}$$

$D_1 = -4$

$D_2 = 8$

$D_3 = -48$

$H(h)$ neg. defin.

⇕

h concave

Summary: Lagrange / Kuhn-Tucker problems

① Find candidate pts:

i) Solve FOC+C / FOC+C + CSC.
$\Rightarrow$ ordinary candidate pts.

ii) Admissible pts where NDCQ fails $\Rightarrow$ special cand. pts.

② Conclude about max/min:

a) If the set D of admissible pts (pts satisfying all constraints) is bounded, use Extreme Value Thm (EVT).

b) If you have a ordinary candidate pt that you think is max/min, try to use the second order condition (SOC).

Sometimes it can be quicker to use the steps in some other order!

Ex: max $f(x,y) = y$ when $x^2 + y^3 = 0$

$\mathcal{L} = y - \lambda \cdot (x^2 + y^3)$

FOC: $\begin{cases} \mathcal{L}'_x = -\lambda \cdot 2x = 0 \\ \mathcal{L}'_y = 1 - \lambda \cdot 3y^2 = 0 \end{cases}$

C: $x^2 + y^3 = 0$

$-2\lambda x = 0$

$\lambda = 0$ or $x = 0$

$\lambda = 0$

$1 = 0$

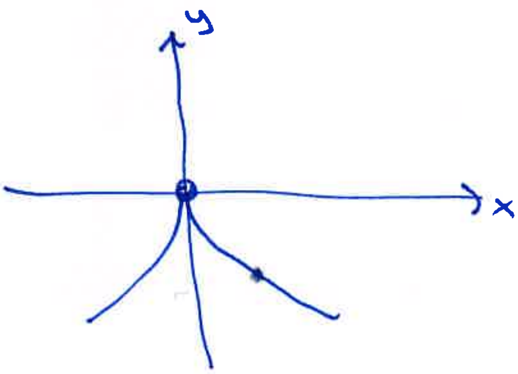
not poss.

$x = 0$

$y^3 = 0 \Rightarrow y = 0$

$1 - 0 = 0$

not possible



no ordinary candidate pts

D: $x^2 + y^3 = 0$

$y^3 = -x^2$

$y = \sqrt[3]{-x^2}$

$= -\sqrt[3]{x^2}$

max $f(x,y) = y$ on $x^2 + y^3 = 0$

max $f = 0$ at $(0,0)$

2 Non-degenerate constraint qualification = NDCQ

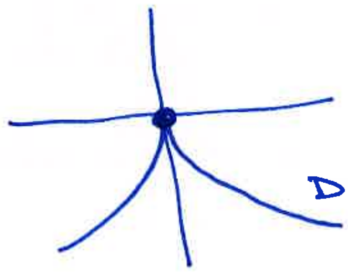
Ex: $g(x,y) = a \qquad J = (g'_x \ \ g'_y)$

Defn: NDCQ holds if $\mathrm{rk}\, J = 1$ (maximal).

$x^2 + y^3 = 0 \ : \ J = (2x \ \ 3y^2)$

$$2x = 0, \quad 3y^2 = 0 \qquad \mathrm{rk}\, J = \begin{cases} 1, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

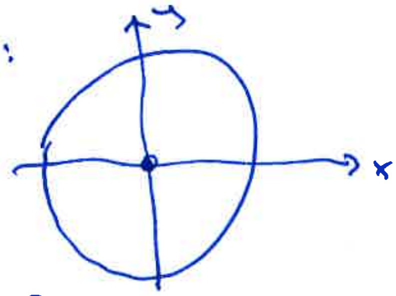
NDCQ fails at $(0,0)$: i) $\mathrm{rk}\, J(0,0) = 0 < 1$
ii) $(0,0)$ admissible $(0^2 + 0^3 = 0)$



$D: x^2 + y^3 = 0$

Admissible points where NDCQ fails are called special candidate pts; these point can be max/min even if they don't satisfy FOC+C.

Ex:



$D: x^2 + y^2 = 4$

no special cond. pts

$J = (2x \ \ 2y)$

$$\mathrm{rk}\, J = \begin{cases} 1, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

$$\left.\begin{matrix} 2x = 0 \\ 2y = 0 \end{matrix}\right\} (0,0) \leftarrow \text{not admissible}$$

NDCQ:

Lagrange:
$$\begin{cases} g_1(x) = a_1 \\ \vdots \\ g_m(\underline{x}) = a_m \end{cases} \qquad J = \begin{pmatrix} \frac{\partial g_1}{\partial x_1} & \frac{\partial g_1}{\partial x_2} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial g_m}{\partial x_1} & \frac{\partial g_m}{\partial x_2} & \cdots & \frac{\partial g_m}{\partial x_n} \end{pmatrix}$$

NDCQ: $\operatorname{rk} J = m$

NDCQ fails: All $\underbrace{m\text{-minors}}_{\text{maximal minors}}$ are zero.

same as

Kuhn-Tucker:
$$\begin{matrix} g_1(\underline{x}) \le a_1 \\ \vdots \\ g_m(\underline{x}) \le a_m \end{matrix} \qquad J = (\;\cdots\;)$$

NDCQ: At a pt $\underline{x}^*$, $\operatorname{rk} J^*$ should be maximal where J^* contain the rows of J corresponding to constraints that are binding ($g_i(\underline{x}^*) = a_i$ equality!)

Ex:
$$\begin{matrix} x - y + 2z \ge 3 \\ x + y \ge 3 \end{matrix} \qquad J = \begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix}$$

a) $x - y + 2z = 3$, $x + y = 3$

NDCQ: $\operatorname{rk}\begin{pmatrix} 1 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix} = 2$

ok.

b) $x - y + 2z = 3$, $x + y > 3$

$\operatorname{rk}(1 \;\; -1 \;\; 2) = 1$

ok

c) $x - y + 2z > 3$, $x + y = 3$

$\operatorname{rk}(1 \;\; 1 \;\; 0) = 1$

ok.

d) $x - y + 2z > 3$, $x + y > 3$

no condition

ok.

NDCQ satisfied in all cases $\Rightarrow$ no special cand. pt.